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On the Prime Labeling of Generalized Petersen Graphs $P(n,3)$ [1]

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Abstract

A graph G with vertex set V is said to have a prime labeling if its vertices can be labeled with distinct integers $1, 2, \ldots, |V|$ such that for every edge xy in E, the labels assigned to x and y are relatively prime or coprime. A graph is called prime if it has a prime labeling. In this paper, we show that generalized Petersen graphs $P(n,3)$ are not prime for odd n, prime for even $n \leq 100$ and conjectured that $P(n,3)$ are prime for all even n.

Mathematics Subject Classification: Primary 05C78; Secondary 52C30

Keywords: Prime labeling, Prime graph, generalized Petersen graph

1 Introduction

We consider only finite undirected graphs without loops or multiple edges. Let $G = (V, E)$ be a graph with vertex set V and edge set E.

A graph G is said to have a prime labeling if its vertices can be labeled with distinct integers $1, 2, \ldots, |V|$ such that for every edge xy in E, the labels assigned to x and y are relatively prime or coprime. A graph is called prime

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if it has a prime labeling. This concept was originated with Entringer and introduced by Tout, Dabboucy, and Howalla[10].

Roger Entringer conjectured that all trees are prime. Fu and Huang [2] proved that every tree with $n \leq 15$ vertices is prime. O'Pikhurko [6, 7] extended this result to all $n \leq 50$. Other prime graphs include all cycles and the disjoint union of C_{2k} and C_n. Seoud, Diab, and Elsakhawi [8] showed that following graphs are prime: Fans, Helms, Flowers, Stars, $K_{2,n}$ and $K_{3,n}$ unless $n = 3$ or 7. They also showed that $P_n + \bar{K}_m$ $(m \geq 3)$ is not prime. Kelli Carlson [1] proved that generalized Books and C_m-Snakes are prime graphs. Vilfred, Somasundaram, and Nicholas [11] have conjectured that the grid $P_m \times P_n$ is prime when n is prime and $n > m$. This conjecture was proved by Sundaram, Ponraj, and Somasundaram [9]. In the same article they also showed that $P_n \times P_n$ is prime when n is prime. The authors [4, 5] proved that the following graphs are prime: generalized Petersen graph $P(n, 1)$ for even $n \leq 2500$ and not prime for odd n and Knödel graphs $W_{3,n}$ for $n \leq 130$. We refer the readers to the dynamic survey by Gallian [3].

The generalized Petersen graphs $P(n, k)$ are defined to be a graph on $2n(n \geq 3)$ vertices with $V(P(n, k)) = \{v_i, u_i : 0 \leq i \leq n-1\}$ and $E(P(n, k)) = \{v_iv_{i+1}, v_iu_i, u_iu_{i+k} : 0 \leq i \leq n-1, \text{ subscripts modulo } n\}$. In this paper, we show that generalized Petersen graphs $P(n, 3)$ are not prime for odd n, prime for even $n \leq 100$, and conjecture that $P(n, 3)$ are prime for all even n.

2 Main Results

Theorem 2.1. $P(n, 3)$ is not prime for odd n.

Proof. By contradiction. Suppose that $P(n, 3)$ is prime for some odd n, say n_1. Let f be a prime labeling of $P(n_1, 3)$. Then one of $\{f(v_0), f(v_1), \ldots, f(v_{n_1-1})\}$ and $\{f(u_0), f(u_1), \ldots, f(u_{n_1-1})\}$ must contains at least $\frac{n_1+1}{2}$ evens, i.e. there are at least two evens adjacent, a contradiction. □

For even n, let

$$\mathcal{N}_i = \{n : n + i \;\; is \;\; prime\}, \;\; \mathcal{N}_i^* = \{n : 2n + i \;\; is \;\; prime\},$$
$$\mathcal{N} = \bigcup_{-8 \leq i \leq 3} (\mathcal{N}_{2i+1} \bigcap \mathcal{N}_{2i+5} \bigcap \mathcal{N}_{2i+7}),$$
$$\mathcal{N}^* = \bigcup_{-5 \leq i \leq 6} (\mathcal{N}_{4i+1}^* \bigcap \mathcal{N}_{4i+5}^* \bigcap \mathcal{N}_{4i+7}^*).$$

We will prove the following Theorem by Lemmas 2.4 - 2.27.

Theorem 2.2. $P(n, 3)$ is prime for even $n \in \mathcal{N} \bigcup \mathcal{N}^*$.

Observation 2.3. $f(u)$ and $f(v)$ are coprime if they satisfy any one of the following conditions:

(1) $f(u) = 1$ or $f(v) = 1$,
(2) $f(u) = 2$ and $f(v)$ is odd or $f(v) = 2$ and $f(u)$ is odd,
(3) $f(u) + f(v)$ is prime,
(4) $|f(u) - f(v)| = 1$,
(5) $|f(u) - 2f(v)| = 1$,
(6) $|f(u) - 2f(v)| = p_1^{t_1} p_2^{t_2} \dots p_k^{t_k}$ and $f(v) \not\equiv 0 \bmod p_i$ $(1 \le i \le k)$,
(7) $|f(u) - f(v)| = p^t$ is a prime power and $f(u) \not\equiv 0 \bmod p$.

Lemma 2.4. $P(n,3)$ is prime for even $n \in \mathcal{N}^*_{-3} \bigcap \mathcal{N}^*_1 \bigcap \mathcal{N}^*_3$.
Proof. We define the function f as follows:
Let

$$f(v_i) = \begin{cases} i+2, & 0 \le i \le n-4, i \bmod 2 = 0, \\ 2n-i, & 0 \le i \le n-4, i \bmod 2 = 1, \\ n-1, & i = n-3, \\ n, & i = n-2, \\ n+1, & i = n-1, \end{cases} \qquad f(u_i) = \begin{cases} i+1, & 0 \le i \le n-4, i \bmod 2 = 0, \\ 2n-i-1, & 0 \le i \le n-4, i \bmod 2 = 1, \\ n+2, & i = n-3, \\ n+3, & i = n-2, \\ 2n, & i = n-1. \end{cases}$$

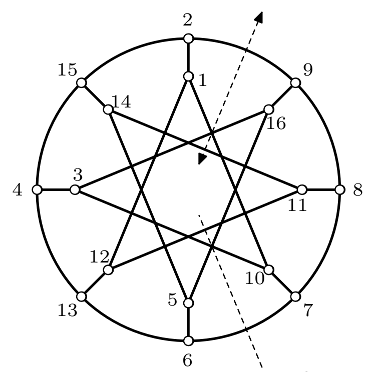


Figure 1: $P(n,3)$ for $n = 8 \in \mathcal{N}^*_{-3} \bigcap \mathcal{N}^*_1 \bigcap \mathcal{N}^*_3$.

In Figure 2.1, we show the prime labeling of $P(n,3)$, where $n = 8 \in \mathcal{N}^*_{-3} \bigcap \mathcal{N}^*_1 \bigcap \mathcal{N}^*_3$.

For $0 \le i \le n-5$, by Observation 2.3(3), $f(v_i)$ and $f(v_{i+1})$ are coprime. For $n-4 \le i \le n-2$, $|f(v_{i+1}) - f(v_i)| = 1$, by Observation 2.3(4), $f(v_i)$ and $f(v_{i+1})$ are coprime. For $i = n-1$, $f(v_0) = 2$ and $f(v_{n-1}) = n+1$ is odd, by Observation 2.2(2), $f(v_{n-1})$ and $f(v_0)$ are coprime.

For $0 \le i \le n-7$, by Observation 2.3(3), $f(u_i)$ and $f(u_{i+3})$ are coprime. For $i = n-6$, $|f(u_{n-6}) - f(u_{n-3})| = |n-5-(n+2)| = 7$, by Observation 2.3(7), $f(u_{n-6})$ and $f(u_{n-3})$ are coprime. For $i = n-5$, $|f(u_{n-5}) - f(u_{n-2})| = |n+4-(n+3)| = 1$, by Observation 2.3(4), $f(u_{n-5})$ and $f(u_{n-2})$ are coprime. For $i = n-4$, $|f(u_{n-1}) - f(u_{n-4})| = |2n - 2(n-3)| = 2 \times 3$. Since $n \in \mathcal{N}^*_{-3} \bigcap \mathcal{N}^*_1 \bigcap \mathcal{N}^*_3$, $n-3 \bmod 3 \ne 0$. Since $n-3$ is odd, we have $f(u_{n-4})$ and $f(u_{n-1})$ are coprime. For $i = n-3$, by Observation 2.3(1), $f(u_{n-3}) = n+2$ and

$f(u_0) = 1$ are coprime. For $i = n-2$, $|f(u_{n-2}) - f(u_{n-7})| = |n+3-(n+6) = 3$, by Observation 2.3(7), $f(u_{n-2})$ and $f(u_{n-7})$ are coprime. For $i = n-1$, $f(u_{n-6}) = 3$, $f(u_{n-1}) = 2n$. Since $n \in \mathcal{N}^*_{-3} \bigcap \mathcal{N}^*_1 \bigcap \mathcal{N}^*_3$, $2n \bmod 3 \neq 0$, hence $f(u_{n-1})$ and $f(u_{n-6})$ are coprime.

For $0 \le i \le n-4$,

$$|f(u_i) - f(v_i)| = \begin{cases} |i+1-(i+2)| = 1, & i \bmod 2 = 0, \\ |2n-i-1-(2n-i)| = 1, & i \bmod 2 = 1, \end{cases}$$

by Observation 2.3(4), $f(v_i)$ and $f(u_i)$ are coprime. For $i = n-3$, $|f(u_{n-3}) - f(v_{n-3})| = |n+2-(n-1)| = 3$, by Observation 2.3(7), $f(v_{n-3})$ and $f(u_{n-3})$ are coprime. For $i = n-2$, $|f(u_{n-2}) - f(v_{n-2})| = |n+3-n| = 3$, by Observation 2.3(7), $f(v_{n-3})$ and $f(u_{n-3})$ are coprime. For $i = n-1$, $|f(u_{n-1}) - 2f(v_{n-1})| = |2n-2(n+1)| = 2$ and $f(v_{n-1})$ is odd, by Observation 2.3(6) $f(v_{n-1})$, $f(u_{n-1})$ are comprime.

Hence f is a prime labeling of $P(n,3)$ for even $n \in \mathcal{N}^*_{-3} \bigcap \mathcal{N}^*_1 \bigcap \mathcal{N}^*_3$. □

For the Lemmas 2.5 - 2.27, we only define f, and leave for the readers to verify that the f is a prime labeling of $P(n,3)$.

Lemma 2.5. *$P(n,3)$ is prime for even $n \in \mathcal{N}^*_{-7} \bigcap \mathcal{N}^*_{-3} \bigcap \mathcal{N}^*_{-1}$.*
Proof. We define the function f as follows:
Let

$$f(v_i) = \begin{cases} i+2, & 0 \le i \le n-4, i \bmod 2 = 0, \\ 2n-i-4, & 0 \le i \le n-4, i \bmod 2 = 1, \\ 2n-3, & i = n-3, \\ 2n-4, & i = n-2, \\ 2n-1, & i = n-1, \end{cases} \quad f(u_i) = \begin{cases} i+1, & 0 \le i \le n-4, i \bmod 2 = 0, \\ 2n-i-5, & 0 \le i \le n-4, i \bmod 2 = 1, \\ 2n-2, & i = n-3, \\ n-1, & i = n-2, \\ 2n, & i = n-1. \end{cases}$$

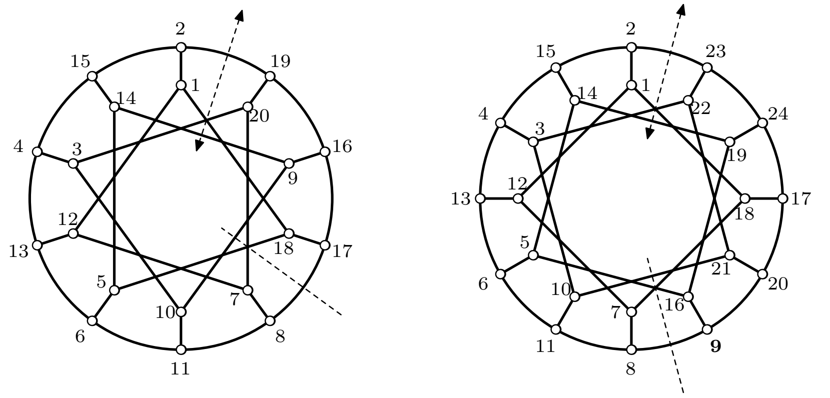
In Figure 2.2(a), we show the prime labeling of $P(n,3)$, where $n = 10 \in \mathcal{N}^*_{-7} \bigcap \mathcal{N}^*_{-3} \bigcap \mathcal{N}^*_{-1}$.

Lemma 2.6. *$P(n,3)$ is prime for even $n \in \mathcal{N}^*_{-11} \bigcap \mathcal{N}^*_{-7} \bigcap \mathcal{N}^*_{-5}$.*
Proof. We define the function f as follows:
Let

$$f(v_i) = \begin{cases} i+2, & 0 \le i \le n-5, i \bmod 2 = 0, \\ 2n-i-8, & 0 \le i \le n-5, i \bmod 2 = 1, \\ 2n-4, & i = n-4, \\ 2n-7, & i = n-3, \\ 2n, & i = n-2, \\ 2n-1, & i = n-1, \end{cases} \quad f(u_i) = \begin{cases} i+1, & 0 \le i \le n-6, i \bmod 2 = 0, \\ 2n-i-9, & 1 \le i \le n-6, i \bmod 2 = 1, \\ 2n-8, & i = n-5, \\ 2n-3, & i = n-4, \\ 2n-6, & i = n-3, \\ 2n-5, & i = n-2, \\ 2n-2, & i = n-1. \end{cases}$$

In Figure 2.2(b), we show the prime labeling of $P(n,3)$, where $n = 12 \in \mathcal{N}^*_{-11} \bigcap \mathcal{N}^*_{-7} \bigcap \mathcal{N}^*_{-5}$.

Lemma 2.7. *$P(n,3)$ is prime for even $n \in \mathcal{N}^*_{-15} \bigcap \mathcal{N}^*_{-11} \bigcap \mathcal{N}^*_{-9}$.*
Proof. We define the function f as follows:



P(10,3), $10 \in \mathcal{N}^*_{-7} \bigcap \mathcal{N}^*_{-3} \bigcap \mathcal{N}^*_{-1}$ P(12,3), $12 \in \mathcal{N}^*_{-11} \bigcap \mathcal{N}^*_{-7} \bigcap \mathcal{N}^*_{-5}$

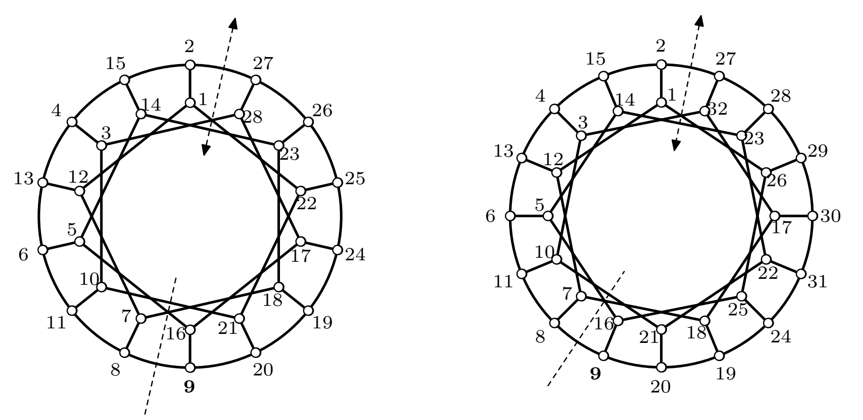
(a) (b)

Figure 2:

Let

$$f(v_i) = \begin{cases} i+2, & 0 \le i \le n-7, i \bmod 2 = 0, \\ 2n-i-12, & 0 \le i \le n-7, i \bmod 2 = 1, \\ 2n-8, & i = n-6, \\ 2n-9, & i = n-5, \\ 2n-4, & i = n-4, \\ 2n-3, & i = n-3, \\ 2n-2, & i = n-2, \\ 2n-1, & i = n-1, \end{cases} \qquad f(u_i) = \begin{cases} i+1, & 0 \le i \le n-8, i \bmod 2 = 0, \\ 2n-i-13, & 0 \le i \le n-8, i \bmod 2 = 1, \\ 2n-12, & i = n-7, \\ 2n-7, & i = n-6, \\ 2n-10, & i = n-5, \\ 2n-11, & i = n-4, \\ 2n-6, & i = n-3, \\ 2n-5, & i = n-2, \\ 2n, & i = n-1. \end{cases}$$

In Figure 2.3(a), we show the prime labeling of $P(n,3)$, where $n = 14 \in \mathcal{N}^*_{-15} \bigcap \mathcal{N}^*_{-11} \bigcap \mathcal{N}^*_{-9}$.



P(14,3), $14 \in \mathcal{N}^*_{-15} \bigcap \mathcal{N}^*_{-11} \bigcap \mathcal{N}^*_{-9}$ P(16,3), $16 \in \mathcal{N}^*_{-19} \bigcap \mathcal{N}^*_{-15} \bigcap \mathcal{N}^*_{-13}$

(a) (b)

Figure 3:

Lemma 2.8. $P(n,3)$ is prime for even $n \in \mathcal{N}^*_{-19} \bigcap \mathcal{N}^*_{-15} \bigcap \mathcal{N}^*_{-13}$.
Proof. We define the function f as follows:

Let

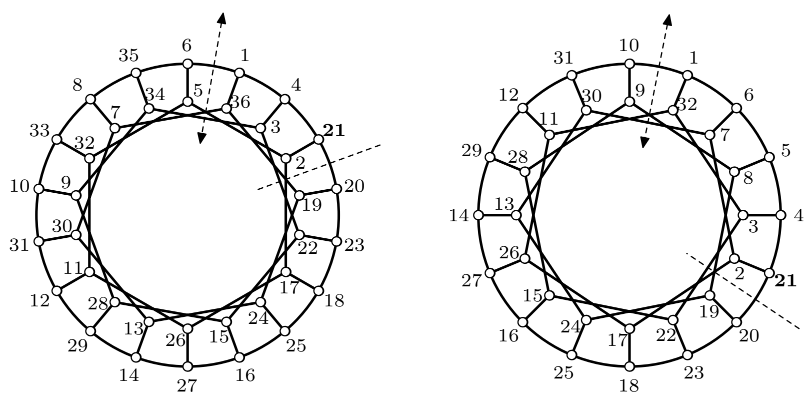
$$f(v_i)=\begin{cases} i+2, & 0\le i\le n-9, i \bmod 2=0,\\ 2n-i-16, & 0\le i\le n-9, i \bmod 2=1,\\ 2n-12, & i=n-8,\\ 2n-13, & i=n-7,\\ 2n-8, & i=n-6,\\ 2n-1, & i=n-5,\\ 2n-2, & i=n-4,\\ 2n-3, & i=n-3,\\ 2n-4, & i=n-2,\\ 2n-5, & i=n-1,\end{cases} \qquad f(u_i)=\begin{cases} i+1, & 0\le i\le n-10, i \bmod 2=0,\\ 2n-i-17, & 0\le i\le n-10, i \bmod 2=1,\\ 2n-16, & i=n-9,\\ 2n-11, & i=n-8,\\ 2n-14, & i=n-7,\\ 2n-7, & i=n-6,\\ 2n-10, & i=n-5,\\ 2n-15, & i=n-4,\\ 2n-6, & i=n-3,\\ 2n-9, & i=n-2,\\ 2n, & i=n-1.\end{cases}$$

In Figure 2.3(b), we show the prime labeling of $P(n,3)$, where $n=16\in\mathcal{N}^*_{-19}\bigcap\mathcal{N}^*_{-15}\bigcap\mathcal{N}^*_{-13}$.

Lemma 2.9. *$P(n,3)$ is prime for even $n\in\mathcal{N}^*_1\bigcap\mathcal{N}^*_5\bigcap\mathcal{N}^*_7$.*
Proof. We define the function f as follows:
Let

$$f(v_i)=\begin{cases} 6+i, & 0\le i\le n-3, i \bmod 2=0,\\ 2n-i, & 0\le i\le n-3, i \bmod 2=1,\\ 4, & i=n-2,\\ 1, & i=n-1,\end{cases} \qquad f(u_i)=\begin{cases} 5+i, & 0\le i\le n-4, i \bmod 2=0,\\ 2n-i-1, & 0\le i\le n-4, i \bmod 2=1,\\ 2, & i=n-3,\\ 3, & i=n-2,\\ 2n, & i=n-1.\end{cases}$$

In Figure 2.4(a), we show the prime labeling of $P(n,3)$, where $n=18\in\mathcal{N}^*_1\bigcap\mathcal{N}^*_5\bigcap\mathcal{N}^*_7$.



P(18, 3), $18\in\mathcal{N}^*_1\bigcap\mathcal{N}^*_5\bigcap\mathcal{N}^*_7$ (a) P(16, 3), $16\in\mathcal{N}^*_5\bigcap\mathcal{N}^*_9\bigcap\mathcal{N}^*_{11}$ (b)

Figure 4:

Lemma 2.10. *$P(n,3)$ is prime for even $n\in\mathcal{N}^*_5\bigcap\mathcal{N}^*_9\bigcap\mathcal{N}^*_{11}$.*
Proof.

In $\mathcal{N}^*_5\bigcap\mathcal{N}^*_9\bigcap\mathcal{N}^*_{11}$, there is only one integer smaller than 16, namely 4. Since $4\in\mathcal{N}^*_{-7}\bigcap\mathcal{N}^*_{-3}\bigcap\mathcal{N}^*_{-1}$, by Lemma 2.5, $P(4,3)$ is prime. Hence, we only consider even $n\ge 16$. And we define the function f as follows:

Let

$$f(v_i)=\begin{cases}10+i, & 0\le i\le n-5, i \bmod 2=0,\\ 2n-i, & 0\le i\le n-5, i \bmod 2=1,\\ 4, & i=n-4,\\ 5, & i=n-3,\\ 6, & i=n-2,\\ 1, & i=n-1,\end{cases} \qquad f(u_i)=\begin{cases}9+i, & 0\le i\le n-6, i \bmod 2=0,\\ 2n-i-1, & 0\le i\le n-6, i \bmod 2=1,\\ 2, & i=n-5,\\ 3, & i=n-4,\\ 8, & i=n-3,\\ 7, & i=n-2,\\ 2n, & i=n-1.\end{cases}$$

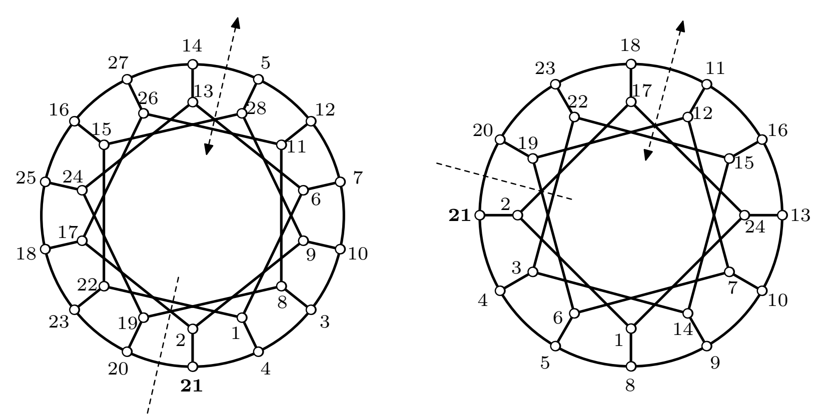
In Figure 2.4(b), we show the prime labeling of $P(n,3)$, where $n=16\in\mathcal{N}_5^*\bigcap\mathcal{N}_9^*\bigcap\mathcal{N}_{11}^*$.

Lemma 2.11. *$P(n,3)$ is prime for even $n\in\mathcal{N}_9^*\bigcap\mathcal{N}_{13}^*\bigcap\mathcal{N}_{15}^*$.*

Proof. We define the function f as follows:
Let

$$f(v_i)=\begin{cases}14+i, & 0\le i\le n-7, i \bmod 2=0,\\ 2n-i, & 0\le i\le n-7, i \bmod 2=1,\\ 4, & i=n-6,\\ 3, & i=n-5,\\ 10, & i=n-4,\\ 7, & i=n-3,\\ 12, & i=n-2,\\ 5, & i=n-1,\end{cases} \qquad f(u_i)=\begin{cases}13+i, & 0\le i\le n-8, i \bmod 2=0,\\ 2n-i-1, & 0\le i\le n-8, i \bmod 2=1,\\ 2, & i=n-7,\\ 1, & i=n-6,\\ 8, & i=n-5,\\ 9, & i=n-4,\\ 6, & i=n-3,\\ 11, & i=n-2,\\ 2n, & i=n-1.\end{cases}$$

In Figure 2.5(a), we show the prime labeling of $P(n,3)$, where $n=14\in\mathcal{N}_9^*\bigcap\mathcal{N}_{13}^*\bigcap\mathcal{N}_{15}^*$.



P(14,3), $14\in\mathcal{N}_9^*\bigcap\mathcal{N}_{13}^*\bigcap\mathcal{N}_{15}^*$
(*a*)

P(12,3), $12\in\mathcal{N}_{13}^*\bigcap\mathcal{N}_{17}^*\bigcap\mathcal{N}_{19}^*$
(*b*)

Figure 5:

Lemma 2.12. *$P(n,3)$ is prime for even $n\in\mathcal{N}_{13}^*\bigcap\mathcal{N}_{17}^*\bigcap\mathcal{N}_{19}^*$.*

Proof. We define the function f as follows:

Let

$$f(v_i)=\begin{cases}18+i, & 0\le i\le n-9, i \bmod 2=0,\\ 2n-i, & 0\le i\le n-9, i \bmod 2=1,\\ 4, & i=n-8,\\ 5, & i=n-7,\\ 8, & i=n-6,\\ 9, & i=n-5,\\ 10, & i=n-4,\\ 13, & i=n-3,\\ 16, & i=n-2,\\ 11, & i=n-1,\end{cases} \qquad f(u_i)=\begin{cases}17+i, & 0\le i\le n-10, i \bmod 2=0,\\ 2n-i-1, & 0\le i\le n-10, i \bmod 2=1,\\ 2, & i=n-9,\\ 3, & i=n-8,\\ 6, & i=n-7,\\ 1, & i=n-6,\\ 14, & i=n-5,\\ 7, & i=n-4,\\ 2n, & i=n-3,\\ 15, & i=n-2,\\ 12, & i=n-1.\end{cases}$$

In Figure 2.5(b), we show the prime labeling of $P(n,3)$, where $n=12\in \mathcal{N}^*_{13}\bigcap\mathcal{N}^*_{17}\bigcap\mathcal{N}^*_{19}$.

Lemma 2.13. $P(n,3)$ is prime for even $n\in\mathcal{N}^*_{17}\bigcap\mathcal{N}^*_{21}\bigcap\mathcal{N}^*_{23}$.

Proof.

In $\mathcal{N}^*_{17}\bigcap\mathcal{N}^*_{21}\bigcap\mathcal{N}^*_{23}$, there is only one integer smaller than 40, namely 10. Since $10\in\mathcal{N}^*_{-7}\bigcap\mathcal{N}^*_{-3}\bigcap\mathcal{N}^*_{-1}$, by Lemma 2.5, $P(10,3)$ is prime. Hence, we only consider even $n\ge 40$. And we define the function f as follows:

Let

$$f(v_i)=\begin{cases}22+i, & 0\le i\le n-11, i \bmod 2=0,\\ 2n-i, & 0\le i\le n-11, i \bmod 2=1,\\ 4, & i=n-10,\\ 3, & i=n-9,\\ 10, & i=n-8,\\ 9, & i=n-7,\\ 14, & i=n-6,\\ 5, & i=n-5,\\ 12, & i=n-4,\\ 11, & i=n-3,\\ 18, & i=n-2,\\ 13, & i=n-1,\end{cases} \qquad f(u_i)=\begin{cases}21+i, & 0\le i\le n-12, i \bmod 2=0,\\ 2n-i-1, & 0\le i\le n-12, i \bmod 2=1,\\ 2, & i=n-11,\\ 1, & i=n-10,\\ 8, & i=n-9,\\ 7, & i=n-8,\\ 2n, & i=n-7,\\ 15, & i=n-6,\\ 6, & i=n-5,\\ 17, & i=n-4,\\ 16, & i=n-3,\\ 19, & i=n-2,\\ 20, & i=n-1.\end{cases}$$

Lemma 2.14. $P(n,3)$ is prime for even $n\in\mathcal{N}^*_{21}\bigcap\mathcal{N}^*_{25}\bigcap\mathcal{N}^*_{27}$.

Proof.

In $\mathcal{N}^*_{21}\bigcap\mathcal{N}^*_{25}\bigcap\mathcal{N}^*_{27}$, there is only one integer smaller than 38, namely 8. Since $8\in\mathcal{N}^*_{-3}\bigcap\mathcal{N}^*_{1}\bigcap\mathcal{N}^*_{3}$, by Lemma 2.4, $P(8,3)$ is prime. Hence, we only consider even $n\ge 38$. And we define the function f as follows:

Let

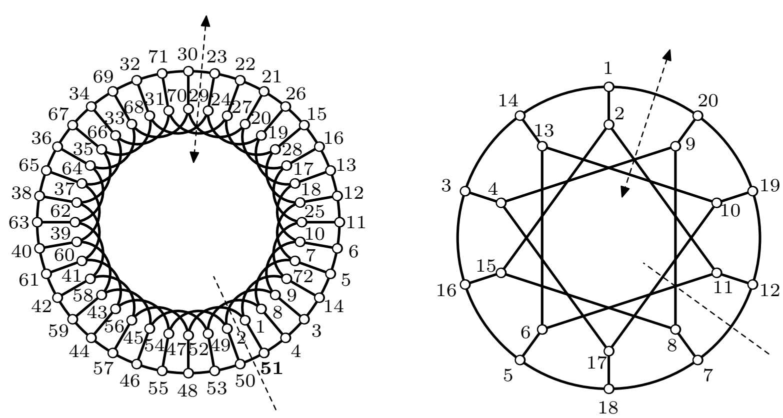
$$f(v_i)=\begin{cases}26+i, & 0\le i\le n-13, i \bmod 2=0,\\ 2n-i, & 0\le i\le n-13, i \bmod 2=1,\\ 4, & i=n-12,\\ 3, & i=n-11,\\ 10, & i=n-10,\\ 9, & i=n-9,\\ 16, & i=n-8,\\ 11, & i=n-7,\\ 12, & i=n-6,\\ 17, & i=n-5,\\ 18, & i=n-4,\\ 19, & i=n-3,\\ 20, & i=n-2,\\ 21, & i=n-1,\end{cases} \qquad f(u_i)=\begin{cases}25+i, & 0\le i\le n-14, i \bmod 2=0,\\ 2n-i-1, & 0\le i\le n-14, i \bmod 2=1,\\ 2, & i=n-13,\\ 1, & i=n-12,\\ 8, & i=n-11,\\ 7, & i=n-10,\\ 14, & i=n-9,\\ 15, & i=n-8,\\ 6, & i=n-7,\\ 13, & i=n-6,\\ 22, & i=n-5,\\ 5, & i=n-4,\\ 24, & i=n-3,\\ 23, & i=n-2,\\ 2n, & i=n-1.\end{cases}$$

Lemma 2.15. $P(n,3)$ is prime for even $n \in \mathcal{N}_{25}^* \bigcap \mathcal{N}_{29}^* \bigcap \mathcal{N}_{31}^*$.
Proof.

In $\mathcal{N}_{25}^* \bigcap \mathcal{N}_{29}^* \bigcap \mathcal{N}_{31}^*$, there is only one integer smaller than 36, namely 6. Since $6 \in \mathcal{N}_1^* \bigcap \mathcal{N}_5^* \bigcap \mathcal{N}_7^*$, by Lemma 2.9, $P(6,3)$ is prime. Hence, we only consider even $n \geq 36$. And we define the function f as follows:
Let

$$f(v_i) = \begin{cases} 30+i, & 0 \leq i \leq n-15, i \bmod 2 = 0, \\ 2n-i, & 0 \leq i \leq n-15, i \bmod 2 = 1, \\ 4, & i = n-14, \\ 3, & i = n-13, \\ 14, & i = n-12, \\ 5, & i = n-11, \\ 6, & i = n-10, \\ 11, & i = n-9, \\ 12, & i = n-8, \\ 13, & i = n-7, \\ 16, & i = n-6, \\ 15, & i = n-5, \\ 26, & i = n-4, \\ 21, & i = n-3, \\ 22, & i = n-2, \\ 23, & i = n-1, \end{cases} \qquad f(u_i) = \begin{cases} 29+i, & 0 \leq i \leq n-16, i \bmod 2 = 0, \\ 2n-i-1, & 0 \leq i \leq n-16, i \bmod 2 = 1, \\ 2, & i = n-15, \\ 1, & i = n-14, \\ 8, & i = n-13, \\ 9, & i = n-12, \\ 2n, & i = n-11, \\ 7, & i = n-10, \\ 10, & i = n-9, \\ 25, & i = n-8, \\ 18, & i = n-7, \\ 17, & i = n-6, \\ 28, & i = n-5, \\ 19, & i = n-4, \\ 20, & i = n-3, \\ 27, & i = n-2, \\ 24, & i = n-1. \end{cases}$$

In Figure 2.6(a) we show the prime labeling of $P(n,3)$ for even $n = 36 \in \mathcal{N}_{25}^* \bigcap \mathcal{N}_{29}^* \bigcap \mathcal{N}_{31}^*$.



P(36, 3), $36 \in \mathcal{N}_{25}^* \bigcap \mathcal{N}_{29}^* \bigcap \mathcal{N}_{31}^*$ (a)

P(10, 3), $10 \in \mathcal{N}_{-3} \bigcap \mathcal{N}_1 \bigcap \mathcal{N}_3$ (b)

Figure 6:

Lemma 2.16. $P(n,3)$ is prime for even $n \in \mathcal{N}_{-3} \bigcap \mathcal{N}_1 \bigcap \mathcal{N}_3$.
Proof. We define the function f as follows:
Let

$$f(v_i) = \begin{cases} i+1, & 0 \leq i \leq n-4, i \bmod 2 = 0, \\ n+i+3, & 0 \leq i \leq n-4, i \bmod 2 = 1, \\ n+2, & i = n-3, \\ 2n-1, & i = n-2, \\ 2n, & i = n-1, \end{cases} \qquad f(u_i) = \begin{cases} i+2, & 0 \leq i \leq n-4, i \bmod 2 = 0, \\ n+i+2, & 0 \leq i \leq n-4, i \bmod 2 = 1, \\ n+1, & i = n-3, \\ n, & n-2, \\ n-1, & i = n-1. \end{cases}$$

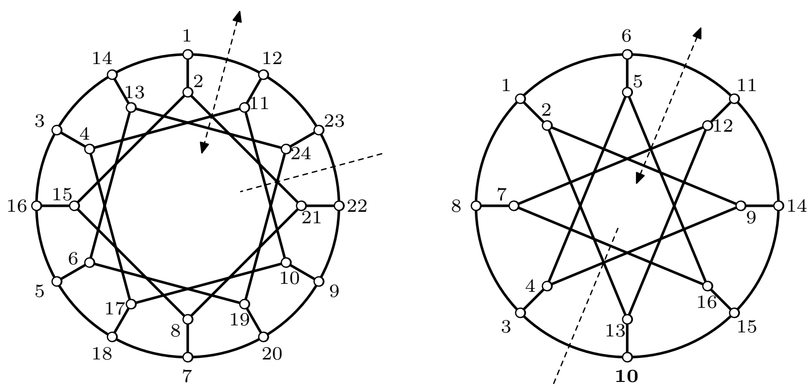
In Figure 2.6(b) we show the prime labeling of $P(n,3)$ for even $n = 10 \in \mathcal{N}_{-3} \bigcap \mathcal{N}_1 \bigcap \mathcal{N}_3$.

Lemma 2.17. $P(n,3)$ is prime for even $n \in \mathcal{N}_{-5} \bigcap \mathcal{N}_{-1} \bigcap \mathcal{N}_1$.

Proof. We define the function f as follows:
Let

$$f(v_i) = \begin{cases} i+1, & 0 \le i \le n-3, i \bmod 2 = 0, \\ n+i+1, & 0 \le i \le n-3, i \bmod 2 = 1, \\ 2n-1, & i = n-2, \\ n, & i = n-1, \end{cases} \qquad f(u_i) = \begin{cases} i+2, & 0 \le i \le n-3, i \bmod 2 = 0, \\ n+i, & 0 \le i \le n-3, i \bmod 2 = 1, \\ 2n, & i = n-2, \\ n-1, & i = n-1. \end{cases}$$

In Figure 2.7(a), we show the prime labeling of $P(n,3)$, where $n = 12 \in \mathcal{N}_{-5} \bigcap \mathcal{N}_{-1} \bigcap \mathcal{N}_1$.



P(12, 3), $12 \in \mathcal{N}_{-5} \bigcap \mathcal{N}_{-1} \bigcap \mathcal{N}_1$ P(8, 3), $8 \in \mathcal{N}_{-7} \bigcap \mathcal{N}_{-3} \bigcap \mathcal{N}_{-1}$

(a) (b)

Figure 7:

Lemma 2.18. $P(n,3)$ is prime for even $n \in \mathcal{N}_{-7} \bigcap \mathcal{N}_{-3} \bigcap \mathcal{N}_{-1}$.

Proof. We define the function f as follows:
Let

$$f(v_i) = \begin{cases} n+i-2, & 0 \le i \le n-4, i \bmod 2 = 0, \\ i, & 0 \le i \le n-4, i \bmod 2 = 1, \\ 2n-1, & i = n-3, \\ 2n-2, & i = n-2, \\ 2n-5, & i = n-1, \end{cases} \qquad f(u_i) = \begin{cases} n+i-3, & 0 \le i \le n-5, i \bmod 2 = 0, \\ i+1, & 0 \le i \le n-5, i \bmod 2 = 1, \\ 2n-3, & i = n-4, \\ 2n, & i = n-3, \\ 2n-7, & i = n-2, \\ 2n-4, & i = n-1. \end{cases}$$

In Figure 2.7(b), we show the prime labeling of $P(n,3)$, where $n = 8 \in \mathcal{N}_{-7} \bigcap \mathcal{N}_{-3} \bigcap \mathcal{N}_{-1}$.

Lemma 2.19. $P(n,3)$ is prime for even $n \in \mathcal{N}_{-9} \bigcap \mathcal{N}_{-5} \bigcap \mathcal{N}_{-3}$.

Proof.

In $\mathcal{N}_{-9} \bigcap \mathcal{N}_{-5} \bigcap \mathcal{N}_{-3}$, there is only one integer smaller than 22, namely 16. Since $16 \in \mathcal{N}_{-3} \bigcap \mathcal{N}_1 \bigcap \mathcal{N}_3$, by Lemma 2.16, $P(16,3)$ is prime. Hence, we only consider even $n \ge 22$. And we define the function f as follows:

Let

$$f(v_i)=\begin{cases} n+i-4, & 0\le i\le n-6, i \bmod 2=0,\\ i, & 0\le i\le n-6, i \bmod 2=1,\\ 2n-5, & i=n-5,\\ 2n, & i=n-4,\\ 2n-3, & i=n-3,\\ 2n-4, & i=n-2,\\ 2n-7, & i=n-1,\end{cases} \qquad f(u_i)=\begin{cases} n+i-5, & 0\le i\le n-7, i \bmod 2=0,\\ i+1, & 0\le i\le n-7, i \bmod 2=1,\\ 2n-1, & i=n-6,\\ 2n-6, & i=n-5,\\ 2n-9, & i=n-4,\\ 2n-2, & i=n-3,\\ 2n-11, & i=n-2,\\ 2n-8, & i=n-1.\end{cases}$$

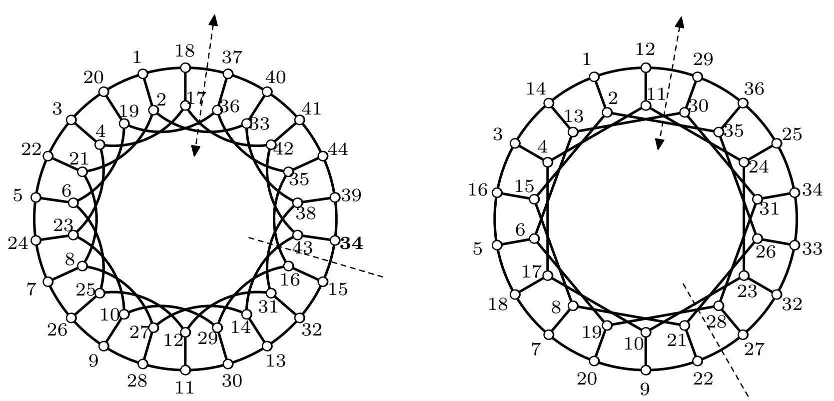
In Figure 2.8(a), we show the prime labeling of $P(n,3)$, where $n=22\in\mathcal{N}_{-9}\bigcap\mathcal{N}_{-5}\bigcap\mathcal{N}_{-3}$.

Lemma 2.20. $P(n,3)$ is prime for even $n\in\mathcal{N}_{-11}\bigcap\mathcal{N}_{-7}\bigcap\mathcal{N}_{-5}$.

Proof. We define the function f as follows:
Let

$$f(v_i)=\begin{cases} n+i-6, & 0\le i\le n-8, i \bmod 2=0,\\ i, & 0\le i\le n-8, i \bmod 2=1,\\ 2n-9, & i=n-7,\\ 2n-4, & i=n-6,\\ 2n-3, & i=n-5,\\ 2n-2, & i=n-4,\\ 2n-11, & i=n-3,\\ 2n, & i=n-2,\\ 2n-7, & i=n-1,\end{cases} \qquad f(u_i)=\begin{cases} n+i-7, & 0\le i\le n-8, i \bmod 2=0,\\ i+1, & 0\le i\le n-8, i \bmod 2=1,\\ 2n-8, & i=n-7,\\ 2n-13, & i=n-6,\\ 2n-10, & i=n-5,\\ 2n-5, & i=n-4,\\ 2n-12, & i=n-3,\\ 2n-1, & i=n-2,\\ 2n-6, & i=n-1.\end{cases}$$

In Figure 2.8(b), we show the prime labeling of $P(n,3)$, where $n=18\in\mathcal{N}_{-11}\bigcap\mathcal{N}_{-7}\bigcap\mathcal{N}_{-5}$.



P(22, 3), $22\in\mathcal{N}_{-9}\bigcap\mathcal{N}_{-5}\bigcap\mathcal{N}_{-3}$
(*a*)

P(18, 3), $18\in\mathcal{N}_{-11}\bigcap\mathcal{N}_{-7}\bigcap\mathcal{N}_{-5}$
(*b*)

Figure 8:

Lemma 2.21. $P(n,3)$ is prime for even $n\in\mathcal{N}_{-13}\bigcap\mathcal{N}_{-9}\bigcap\mathcal{N}_{-7}$.

Proof.

In $\mathcal{N}_{-13}\bigcap\mathcal{N}_{-9}\bigcap\mathcal{N}_{-7}$, there is only one integer smaller than 26, namely 20. Since $20\in\mathcal{N}_{-7}\bigcap\mathcal{N}_{-3}\bigcap\mathcal{N}_{-1}$, by Lemma 2.18, $P(20,3)$ is prime. Hence, we only consider even $n\geq 26$. And we define the function f as follows:

Case 1. $n\equiv 1 \bmod 7$. Let

$$f(v_i)=\begin{cases} n+i-8, & 0\le i\le n-10, i \bmod 2=0,\\ i, & 0\le i\le n-10, i \bmod 2=1,\\ 2n-17, & i=n-9,\\ 2n-14, & i=n-8,\\ 2n-9, & i=n-7,\\ 2n-10, & i=n-6,\\ 2n-3, & i=n-5,\\ 2n-8, & i=n-4,\\ 2n-1, & i=n-3,\\ 2n-6, & i=n-2,\\ 2n-15, & i=n-1,\end{cases} \qquad f(u_i)=\begin{cases} n+i-9, & 0\le i\le n-10, i \bmod 2=0,\\ i+1, & 0\le i\le n-10, i \bmod 2=1,\\ 2n-16, & i=n-9,\\ 2n-13, & i=n-8,\\ 2n-12, & i=n-7,\\ 2n-11, & i=n-6,\\ 2n, & i=n-5,\\ 2n-5, & i=n-4,\\ 2n-2, & i=n-3,\\ 2n-7, & i=n-2,\\ 2n-4, & i=n-1.\end{cases}$$

Case 2. $n\equiv 3 \bmod 7$. Let

$$f(v_i)=\begin{cases} n+i-8, & 0\le i\le n-10, i \bmod 2=0,\\ i, & 0\le i\le n-10, i \bmod 2=1,\\ 2n-17, & i=n-9,\\ 2n-14, & i=n-8,\\ 2n-13, & i=n-7,\\ 2n, & i=n-6,\\ 2n-1, & i=n-5,\\ 2n-2, & i=n-4,\\ 2n-3, & i=n-3,\\ 2n-6, & i=n-2,\\ 2n-15, & i=n-1,\end{cases} \qquad f(u_i)=\begin{cases} n+i-9, & 0\le i\le n-10, i \bmod 2=0,\\ i+1, & 0\le i\le n-10, i \bmod 2=1,\\ 2n-16, & i=n-9,\\ 2n-11, & i=n-8,\\ 2n-12, & i=n-7,\\ 2n-9, & i=n-6,\\ 2n-8, & i=n-5,\\ 2n-5, & i=n-4,\\ 2n-10, & i=n-3,\\ 2n-7, & i=n-2,\\ 2n-4, & i=n-1.\end{cases}$$

Case 3. $n\equiv 4 \bmod 7$. Let

$$f(v_i)=\begin{cases} n+i-8, & 0\le i\le n-10, i \bmod 2=0,\\ i, & 0\le i\le n-10, i \bmod 2=1,\\ 2n-17, & i=n-9,\\ 2n-14, & i=n-8,\\ 2n-15, & i=n-7,\\ 2n-6, & i=n-6,\\ 2n-7, & i=n-5,\\ 2n-8, & i=n-4,\\ 2n-3, & i=n-3,\\ 2n-2, & i=n-2,\\ 2n-9, & i=n-1,\end{cases} \qquad f(u_i)=\begin{cases} n+i-9, & 0\le i\le n-10, i \bmod 2=0,\\ i+1, & 0\le i\le n-10, i \bmod 2=1,\\ 2n-16, & i=n-9,\\ 2n-13, & i=n-8,\\ 2n-12, & i=n-7,\\ 2n-11, & i=n-6,\\ 2n, & i=n-5,\\ 2n-5, & i=n-4,\\ 2n-10, & i=n-3,\\ 2n-1, & i=n-2,\\ 2n-4, & i=n-1.\end{cases}$$

Case 4. $n\not\equiv 1,3,4 \bmod 7$. Let

$$f(v_i)=\begin{cases} n+i-8, & 0\le i\le n-10, i \bmod 2=0,\\ i, & 0\le i\le n-10, i \bmod 2=1,\\ 2n-17, & i=n-9,\\ 2n-14, & i=n-8,\\ 2n-7, & i=n-7,\\ 2n-8, & i=n-6,\\ 2n-5, & i=n-5,\\ 2n-12, & i=n-4,\\ 2n-3, & i=n-3,\\ 2n-2, & i=n-2,\\ 2n-11, & i=n-1,\end{cases} \qquad f(u_i)=\begin{cases} n+i-9, & 0\le i\le n-10, i \bmod 2=0,\\ i+1, & 0\le i\le n-10, i \bmod 2=1,\\ 2n-16, & i=n-9,\\ 2n-13, & i=n-8,\\ 2n, & i=n-7,\\ 2n-15, & i=n-6,\\ 2n-6, & i=n-5,\\ 2n-9, & i=n-4,\\ 2n-4, & i=n-3,\\ 2n-1, & i=n-2,\\ 2n-10, & i=n-1.\end{cases}$$

In Figure 2.9(a), we show the prime labeling of $P(n,3)$, where $n=26\in \mathcal{N}_{-13}\bigcap\mathcal{N}_{-9}\bigcap\mathcal{N}_{-7}$.

Lemma 2.22. $P(n,3)$ is prime for even $n \in \mathcal{N}_{-15} \bigcap \mathcal{N}_{-11} \bigcap \mathcal{N}_{-9}$.
Proof.

In $\mathcal{N}_{-15} \bigcap \mathcal{N}_{-11} \bigcap \mathcal{N}_{-9}$, there are only two integers smaller than 52, namely 22,28. Since $22 \in \mathcal{N}_{-9} \bigcap \mathcal{N}_{-5} \bigcap \mathcal{N}_{-3}$, by Lemma 2.19, $P(22,3)$ is prime. Since $28 \in \mathcal{N}^*_{-19} \bigcap \mathcal{N}^*_{-15} \bigcap \mathcal{N}^*_{-13}$, by Lemma 2.8, $P(28,3)$ is prime. Hence, we only consider even $n \geq 52$. And we define the function f as follows:
Case 1. $n \equiv 0 \bmod 7$. Let

$$f(v_i) = \begin{cases} n+i-10, & 0 \leq i \leq n-12, i \bmod 2 = 0, \\ i, & 0 \leq i \leq n-12, i \bmod 2 = 1, \\ 2n-1, & i = n-11, \\ 2n-2, & i = n-10, \\ 2n-7, & i = n-9, \\ 2n-6, & i = n-8, \\ 2n-9, & i = n-7, \\ 2n-12, & i = n-6, \\ 2n-21, & i = n-5, \\ 2n-20, & i = n-4, \\ 2n-19, & i = n-3, \\ 2n-10, & i = n-2, \\ 2n-13, & i = n-1, \end{cases} \quad f(u_i) = \begin{cases} n+i-11, & 0 \leq i \leq n-12, i \bmod 2 = 0, \\ i+1, & 1 \leq i \leq n-12, i \bmod 2 = 1, \\ 2n, & i = n-11, \\ 2n-3, & i = n-10, \\ 2n-4, & i = n-9, \\ 2n-5, & i = n-8, \\ 2n-8, & i = n-7, \\ 2n-17, & i = n-6, \\ 2n-18, & i = n-5, \\ 2n-15, & i = n-4, \\ 2n-16, & i = n-3, \\ 2n-11, & i = n-2, \\ 2n-14, & i = n-1. \end{cases}$$

Case 2. $n \equiv 3 \bmod 7$. Let

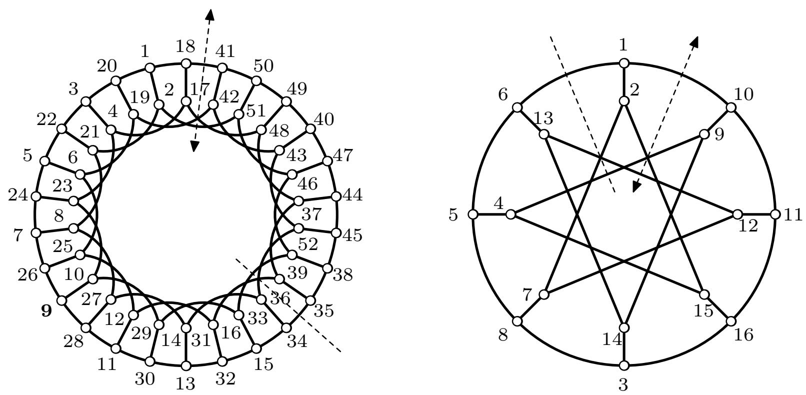
$$f(v_i) = \begin{cases} n-i-10, & 0 \leq i \leq n-12, i \bmod 2 = 0, \\ i, & 0 \leq i \leq n-12, i \bmod 2 = 1, \\ 2n-1, & i = n-11, \\ 2n-2, & i = n-10, \\ 2n-7, & i = n-9, \\ 2n-18, & i = n-8, \\ 2n-5, & i = n-7, \\ 2n-4, & i = n-6, \\ 2n-17, & i = n-5, \\ 2n-10, & i = n-4, \\ 2n-15, & i = n-3, \\ 2n-20, & i = n-2, \\ 2n-19, & i = n-1, \end{cases} \quad f(u_i) = \begin{cases} n-i-11, & 0 \leq i \leq n-12, i \bmod 2 = 0, \\ i+1, & 0 \leq i \leq n-12, i \bmod 2 = 1, \\ 2n, & i = n-11, \\ 2n-3, & i = n-10, \\ 2n-16, & i = n-9, \\ 2n-11, & i = n-8, \\ 2n-6, & i = n-7, \\ 2n-13, & i = n-6, \\ 2n-12, & i = n-5, \\ 2n-9, & i = n-4, \\ 2n-14, & i = n-3, \\ 2n-21, & i = n-2, \\ 2n-8, & i = n-1. \end{cases}$$

Case 3. $n \equiv 5 \bmod 7$. Let

$$f(v_i) = \begin{cases} n-i-10, & 0 \leq i \leq n-12, i \bmod 2 = 0, \\ i, & 0 \leq i \leq n-12, i \bmod 2 = 1, \\ 2n-17, & i = n-11, \\ 2n-12, & i = n-10, \\ 2n-19, & i = n-9, \\ 2n-20, & i = n-8, \\ 2n-13, & i = n-7, \\ 2n-10, & i = n-6, \\ 2n-9, & i = n-5, \\ 2n-6, & i = n-4, \\ 2n-5, & i = n-3, \\ 2n, & i = n-2, \\ 2n-1, & i = n-1, \end{cases} \quad f(u_i) = \begin{cases} n-i-11, & 0 \leq i \leq n-12, i \bmod 2 = 0, \\ i+1, & 0 \leq i \leq n-12, i \bmod 2 = 1, \\ 2n-16, & i = n-11, \\ 2n-21, & i = n-10, \\ 2n-18, & i = n-9, \\ 2n-15, & i = n-8, \\ 2n-14, & i = n-7, \\ 2n-11, & i = n-6, \\ 2n-8, & i = n-5, \\ 2n-7, & i = n-4, \\ 2n-4, & i = n-3, \\ 2n-3, & i = n-2, \\ 2n-2, & i = n-1. \end{cases}$$

Case 4. $n \not\equiv 0,3,5 \bmod 7$, and $n \geq 52$. Let

$$f(v_i)=\begin{cases} n+i-10, & 0\le i\le n-12, i \bmod 2=0,\\ i, & 0\le i\le n-12, i \bmod 2=1,\\ 2n-1, & i=n-11,\\ 2n-2, & i=n-10,\\ 2n-9, & i=n-9,\\ 2n-10, & i=n-8,\\ 2n-5, & i=n-7,\\ 2n-4, & i=n-6,\\ 2n-7, & i=n-5,\\ 2n-14, & i=n-4,\\ 2n-21, & i=n-3,\\ 2n-18, & i=n-2,\\ 2n-19, & i=n-1,\end{cases} \qquad f(u_i)=\begin{cases} n-i-11, & 0\le i\le n-12, i \bmod 2=0,\\ i+1, & 0\le i\le n-12, i \bmod 2=1,\\ 2n, & i=n-11,\\ 2n-3, & i=n-10,\\ 2n-16, & i=n-9,\\ 2n-11, & i=n-8,\\ 2n-6, & i=n-7,\\ 2n-13, & i=n-6,\\ 2n-12, & i=n-5,\\ 2n-15, & i=n-4,\\ 2n-20, & i=n-3,\\ 2n-17, & i=n-2,\\ 2n-8, & i=n-1.\end{cases}$$



P(26, 3), $26 \in \mathcal{N}_{-13}\bigcap\mathcal{N}_{-9}\bigcap\mathcal{N}_{-7}$ (a) P(8, 3), $8 \in \mathcal{N}_{-1}\bigcap\mathcal{N}_{3}\bigcap\mathcal{N}_{5}$ (b)

Figure 9:

Lemma 2.23. $P(n, 3)$ is prime for even $n \in \mathcal{N}_{-1}\bigcap\mathcal{N}_3\bigcap\mathcal{N}_5$.
Proof. We define the function f as follows:
Let

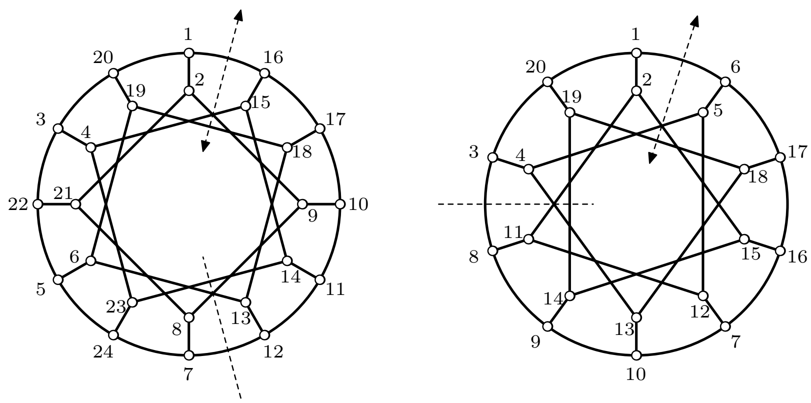
$$f(v_i)=\begin{cases} i+1, & 0\le i\le n-8, i \bmod 2=0,\\ n+i+5, & 0\le i\le n-8, i \bmod 2=1,\\ n-2, & i=n-7,\\ n-3, & i=n-6,\\ n, & i=n-5,\\ n-5, & i=n-4,\\ 2n, & i=n-3,\\ n+3, & i=n-2,\\ n+2, & i=n-1,\end{cases} \qquad f(u_i)=\begin{cases} i+2, & 0\le i\le n-8, i \bmod 2=0,\\ i+n+4, & 0\le i\le n-8, i \bmod 2=1,\\ 2n-3, & i=n-7,\\ n-4, & i=n-6,\\ n-1, & i=n-5,\\ 2n-2, & i=n-4,\\ 2n-1, & i=n-3,\\ n+4, & i=n-2,\\ n+1, & i=n-1.\end{cases}$$

In Figure 2.9(b), we show the prime labeling of $P(n, 3)$, where $n = 8 \in \mathcal{N}_{-1}\bigcap\mathcal{N}_3\bigcap\mathcal{N}_5$.
Lemma 2.24. $P(n, 3)$ is prime for even $n \in \mathcal{N}_1\bigcap\mathcal{N}_5\bigcap\mathcal{N}_7$.
Proof. We define the function f as follows:

Let

$$f(v_i)=\begin{cases} i+1, & 0\le i\le n-6, i \bmod 2=0,\\ n+i+7, & 0\le i\le n-6, i \bmod 2=1,\\ n, & i=n-5,\\ n-1, & i=n-4,\\ n-2, & i=n-3,\\ n+5, & i=n-2,\\ n+4, & i=n-1,\end{cases} \qquad f(u_i)=\begin{cases} i+2, & 0\le i\le n-6, i \bmod 2=0,\\ n+i+6, & 0\le i\le n-6, i \bmod 2=1,\\ n+1, & i=n-5,\\ n+2, & i=n-4,\\ n-3, & i=n-3,\\ n+6, & i=n-2,\\ n+3, & i=n-1.\end{cases}$$

In Figure 2.10(a), we show the prime labeling of $P(n,3)$, where $n=12\in\mathcal{N}_1\bigcap\mathcal{N}_5\bigcap\mathcal{N}_7$.



P(12, 3), $12\in\mathcal{N}_1\bigcap\mathcal{N}_5\bigcap\mathcal{N}_7$ (a) P(10, 3), $10\in\mathcal{N}_3\bigcap\mathcal{N}_7\bigcap\mathcal{N}_9$ (b)

Figure 10:

Lemma 2.25. $P(n,3)$ is prime for even $n\in\mathcal{N}_3\bigcap\mathcal{N}_7\bigcap\mathcal{N}_9$.

Proof.

In $\mathcal{N}_3\bigcap\mathcal{N}_7\bigcap\mathcal{N}_9$, there is only one integer smaller than 10, namely 4. Since $4\in\mathcal{N}_{-3}\bigcap\mathcal{N}_1\bigcap\mathcal{N}_3$, by Lemma 2.16, $P(4,3)$ is prime. Hence, we only consider even $n\ge 10$. And we define the function f as follows:

Let

$$f(v_i)=\begin{cases} i+1, & 0\le i\le n-8, i \bmod 2=0,\\ n+i+9, & 0\le i\le n-8, i \bmod 2=1,\\ n-2, & i=n-7,\\ n-1, & i=n-6,\\ n, & i=n-5,\\ n-3, & i=n-4,\\ n+6, & i=n-3,\\ n+7, & i=n-2,\\ n-4, & i=n-1,\end{cases} \qquad f(u_i)=\begin{cases} i+2, & 0\le i\le n-8, i \bmod 2=0,\\ n+i+8, & 0\le i\le n-8, i \bmod 2=1,\\ n+1, & i=n-7,\\ n+4, & i=n-6,\\ n+3, & i=n-5,\\ n+2, & i=n-4,\\ n+5, & i=n-3,\\ n+8, & i=n-2,\\ n-5, & i=n-1.\end{cases}$$

In Figure 2.10(b), we show the prime labeling of $P(n,3)$, where $n=10\in\mathcal{N}_3\bigcap\mathcal{N}_7\bigcap\mathcal{N}_9$.

Lemma 2.26. $P(n,3)$ is prime for even $n\in\mathcal{N}_5\bigcap\mathcal{N}_9\bigcap\mathcal{N}_{11}$.

Proof.

In $\mathcal{N}_5\bigcap\mathcal{N}_9\bigcap\mathcal{N}_{11}$, there is only one integer smaller than 32, namely 8. Since $8\in\mathcal{N}_{-1}\bigcap\mathcal{N}_3\bigcap\mathcal{N}_5$, by Lemma 2.23, $P(8,3)$ is prime. Hence, we only consider even $n\ge 32$. And we define the function f as follows:

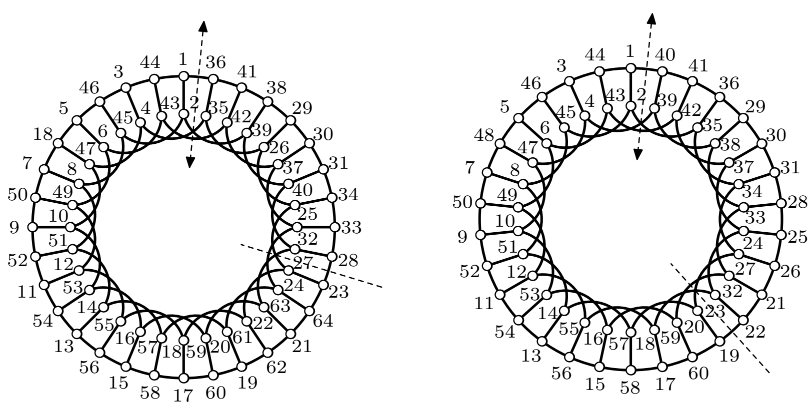
Case 1. $n \equiv 5 \bmod 13$. Let

$$f(v_i) = \begin{cases} i+1, & 0 \le i \le n-10, i \bmod 2 = 0, \\ n+i+11, & 0 \le i \le n-10, i \bmod 2 = 1, \\ n-4, & i = n-9, \\ n-7, & i = n-8, \\ n+2, & i = n-7, \\ n+1, & i = n-6, \\ n+6, & i = n-5, \\ n+5, & i = n-4, \\ n+4, & i = n-3, \\ n-1, & i = n-2, \\ n-2, & i = n-1, \end{cases} \qquad f(u_i) = \begin{cases} i+2, & 0 \le i \le n-10, i \bmod 2 = 0, \\ n+i+10, & 0 \le i \le n-10, i \bmod 2 = 1, \\ n-5, & i = n-9, \\ n-6, & i = n-8, \\ n+9, & i = n-7, \\ n, & i = n-6, \\ n+7, & i = n-5, \\ n+10, & i = n-4, \\ n+3, & i = n-3, \\ n+8, & i = n-2, \\ n-3, & i = n-1. \end{cases}$$

Case 2. $n \not\equiv 5 \bmod 13$. Let

$$f(v_i) = \begin{cases} i+1, & 0 \le i \le n-10, i \bmod 2 = 0, \\ n+i+11, & 0 \le i \le n-10, i \bmod 2 = 1, \\ n-4, & i = n-9, \\ n+1, & i = n-8, \\ n+2, & i = n-7, \\ n-1, & i = n-6, \\ n-2, & i = n-5, \\ n-3, & i = n-4, \\ n+6, & i = n-3, \\ n+9, & i = n-2, \\ n+4, & i = n-1, \end{cases} \qquad f(u_i) = \begin{cases} i+2, & 0 \le i \le n-10, i \bmod 2 = 0, \\ n+i+10, & 0 \le i \le n-10, i \bmod 2 = 1, \\ n-5, & i = n-9, \\ n, & i = n-8, \\ n-7, & i = n-7, \\ n+8, & i = n-6, \\ n+5, & i = n-5, \\ n-6, & i = n-4, \\ n+7, & i = n-3, \\ n+10, & i = n-2, \\ n+3, & i = n-1. \end{cases}$$

In Figure 2.11(a), we show the prime labeling of $P(n,3)$, where $n = 32 \in \mathcal{N}_5 \bigcap \mathcal{N}_9 \bigcap \mathcal{N}_{11}$.



P(32, 3), $32 \in \mathcal{N}_5 \bigcap \mathcal{N}_9 \bigcap \mathcal{N}_{11}$
(a)

P(30, 3), $30 \in \mathcal{N}_7 \bigcap \mathcal{N}_{11} \bigcap \mathcal{N}_{13}$
(b)

Figure 11:

Lemma 2.27. $P(n,3)$ is prime for even $n \in \mathcal{N}_7 \bigcap \mathcal{N}_{11} \bigcap \mathcal{N}_{13}$.
Proof.

In $\mathcal{N}_7 \bigcap \mathcal{N}_{11} \bigcap \mathcal{N}_{13}$, there is only one integer smaller than 30, namely 6. Since $6 \in \mathcal{N}_1 \bigcap \mathcal{N}_5 \bigcap \mathcal{N}_7$, by Lemma 2.24, $P(6,3)$ is prime. Hence, we only consider even $n \ge 30$. And we define the function f as follows:

Table 2.1.

$n \in \mathcal{N}_i \bigcap \mathcal{N}_{i+4} \bigcap \mathcal{N}_{i+6}$	$n \in \mathcal{N}_i \bigcap \mathcal{N}_{i+4} \bigcap \mathcal{N}_{i+6}$	$n \in \mathcal{N}_i \bigcap \mathcal{N}_{i+4} \bigcap \mathcal{N}_{i+6}$	$n \in \mathcal{N}_i \bigcap \mathcal{N}_{i+4} \bigcap \mathcal{N}_{i+6}$
$4 \in \mathcal{N}_{-3} \bigcap \mathcal{N}_1 \bigcap \mathcal{N}_3$	$30 \in \mathcal{N}_7 \bigcap \mathcal{N}_{11} \bigcap \mathcal{N}_{13}$	$56 \in {\mathcal{N}^*}_{-15} \bigcap {\mathcal{N}^*}_{-11} \bigcap {\mathcal{N}^*}_{-9}$	$82 \in \mathcal{N}_{-15} \bigcap \mathcal{N}_{-11} \bigcap \mathcal{N}_{-9}$
$6 \in \mathcal{N}_{-5} \bigcap \mathcal{N}_{-1} \bigcap \mathcal{N}_1$	$32 \in \mathcal{N}_5 \bigcap \mathcal{N}_9 \bigcap \mathcal{N}_{11}$	$58 \in {\mathcal{N}^*}_{-19} \bigcap {\mathcal{N}^*}_{-15} \bigcap {\mathcal{N}^*}_{-13}$	$84 \in \mathcal{N}^*_{25} \bigcap \mathcal{N}^*_{29} \bigcap \mathcal{N}^*_{31}$
$8 \in \mathcal{N}_{-7} \bigcap \mathcal{N}_{-3} \bigcap \mathcal{N}_{-1}$	$34 \in \mathcal{N}_3 \bigcap \mathcal{N}_7 \bigcap \mathcal{N}_9$	$60 \in \mathcal{N}_7 \bigcap \mathcal{N}_{11} \bigcap \mathcal{N}_{13}$	$86 \in \mathcal{N}^*_{21} \bigcap \mathcal{N}^*_{15} \bigcap \mathcal{N}^*_{11}$
$10 \in \mathcal{N}_{-3} \bigcap \mathcal{N}_1 \bigcap \mathcal{N}_3$	$36 \in \mathcal{N}_1 \bigcap \mathcal{N}_5 \bigcap \mathcal{N}_7$	$62 \in \mathcal{N}_5 \bigcap \mathcal{N}_9 \bigcap \mathcal{N}_{11}$	$88 \in \mathcal{N}^*_{17} \bigcap \mathcal{N}^*_{21} \bigcap \mathcal{N}^*_{23}$
$12 \in \mathcal{N}_{-5} \bigcap \mathcal{N}_{-1} \bigcap \mathcal{N}_1$	$38 \in \mathcal{N}_{-1} \bigcap \mathcal{N}_3 \bigcap \mathcal{N}_5$	$64 \in \mathcal{N}_3 \bigcap \mathcal{N}_7 \bigcap \mathcal{N}_9$	$90 \in \mathcal{N}_7 \bigcap \mathcal{N}_{11} \bigcap \mathcal{N}_{13}$
$14 \in \mathcal{N}_{-7} \bigcap \mathcal{N}_{-3} \bigcap \mathcal{N}_{-1}$	$40 \in \mathcal{N}_{-3} \bigcap \mathcal{N}_1 \bigcap \mathcal{N}_3$	$66 \in \mathcal{N}_1 \bigcap \mathcal{N}_5 \bigcap \mathcal{N}_7$	$92 \in \mathcal{N}_5 \bigcap \mathcal{N}_9 \bigcap \mathcal{N}_{11}$
$16 \in \mathcal{N}_{-3} \bigcap \mathcal{N}_1 \bigcap \mathcal{N}_3$	$42 \in \mathcal{N}_{-5} \bigcap \mathcal{N}_{-1} \bigcap \mathcal{N}_1$	$68 \in \mathcal{N}_{-1} \bigcap \mathcal{N}_3 \bigcap \mathcal{N}_5$	$94 \in \mathcal{N}_3 \bigcap \mathcal{N}_7 \bigcap \mathcal{N}_9$
$18 \in \mathcal{N}_{-5} \bigcap \mathcal{N}_{-1} \bigcap \mathcal{N}_1$	$44 \in \mathcal{N}_{-7} \bigcap \mathcal{N}_{-3} \bigcap \mathcal{N}_{-1}$	$70 \in \mathcal{N}_{-3} \bigcap \mathcal{N}_1 \bigcap \mathcal{N}_3$	$96 \in \mathcal{N}_1 \bigcap \mathcal{N}_5 \bigcap \mathcal{N}_7$
$20 \in \mathcal{N}_{-7} \bigcap \mathcal{N}_{-3} \bigcap \mathcal{N}_{-1}$	$46 \in \mathcal{N}_{-9} \bigcap \mathcal{N}_{-4} \bigcap \mathcal{N}_{-3}$	$72 \in \mathcal{N}_{-5} \bigcap \mathcal{N}_{-1} \bigcap \mathcal{N}_1$	$98 \in \mathcal{N}_{-1} \bigcap \mathcal{N}_3 \bigcap \mathcal{N}_5$
$22 \in \mathcal{N}_{-9} \bigcap \mathcal{N}_{-5} \bigcap \mathcal{N}_{-3}$	$48 \in \mathcal{N}_{-11} \bigcap \mathcal{N}_{-7} \bigcap \mathcal{N}_{-5}$	$74 \in \mathcal{N}_{-7} \bigcap \mathcal{N}_{-3} \bigcap \mathcal{N}_{-1}$	$100 \in \mathcal{N}_{-3} \bigcap \mathcal{N}_{-1} \bigcap \mathcal{N}_3$
$24 \in \mathcal{N}_{-11} \bigcap \mathcal{N}_{-7} \bigcap \mathcal{N}_{-5}$	$50 \in \mathcal{N}_{-13} \bigcap \mathcal{N}_{-9} \bigcap \mathcal{N}_{-3}$	$76 \in \mathcal{N}_{-9} \bigcap \mathcal{N}_{-5} \bigcap \mathcal{N}_{-3}$	
$26 \in \mathcal{N}_{-13} \cap \mathcal{N}_{-9} \cap \mathcal{N}_{-7}$	$52 \in \mathcal{N}_{-15} \cap \mathcal{N}_{-11} \cap \mathcal{N}_{-9}$	$78 \in \mathcal{N}_{-11} \cap \mathcal{N}_{-7} \cap \mathcal{N}_{-5}$	
$28 \in \mathcal{N}_{-15} \cap \mathcal{N}_{-11} \cap \mathcal{N}_{-9}$	$54 \in {\mathcal{N}^*}_{-11} \cap {\mathcal{N}^*}_{-7} \cap {\mathcal{N}^*}_{-5}$	$80 \in \mathcal{N}_{-13} \cap \mathcal{N}_{-9} \cap \mathcal{N}_{-7}$	

Let

$$f(v_i) = \begin{cases} i+1, & 0 \le i \le n-12, i \bmod 2 = 0, \\ n+i+13, & 0 \le i \le n-12, i \bmod 2 = 1, \\ n-8, & i = n-11, \\ n-9, & i = n-10, \\ n-4, & i = n-9, \\ n-5, & i = n-8, \\ n-2, & i = n-7, \\ n+1, & i = n-6, \\ n, & i = n-5, \\ n-1, & i = n-4, \\ n+6, & i = n-3, \\ n+11, & i = n-2, \\ n+10, & i = n-1, \end{cases} \qquad f(u_i) = \begin{cases} i+2, & 0 \le i \le n-12, i \bmod 2 = 0, \\ n+i+12, & 0 \le i \le n-12, i \bmod 2 = 1, \\ n-7, & i = n-11, \\ n+2, & i = n-10, \\ n-3, & i = n-9, \\ n-6, & i = n-8, \\ n+3, & i = n-7, \\ n+4, & i = n-6, \\ n+7, & i = n-5, \\ n+8, & i = n-4, \\ n+5, & i = n-3, \\ n+12, & i = n-2, \\ n+9, & i = n-1. \end{cases}$$

In Figure 2.11(b), we show the prime labeling of $P(n,3)$, where $n = 30 \in \mathcal{N}_7 \bigcap \mathcal{N}_{11} \bigcap \mathcal{N}_{13}$.

From the Lemmas 2.4 - 2.27, Theorem 2.2 holds. Furthermore, we have the following conjecture

Conjecture 2.28. $P(n,3)$ is prime for all even n.

Since $n \in \mathcal{N} \bigcup \mathcal{N}^*$ for any even $n \le 100$, by Theorem 2.2 and Table 2.1, we have Conjecture 2.28 holds for even $n \le 100$.

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