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Density and Independently on $H_{bc}(E)$

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Abstract

The aim of the paper ahead Birhoff, Maclane, Godefroy-Shapiro and Kitai-Getner-Shapiro and the results of their theorems and hypercyclic operators on space $H(C)$. In Birhoffs theorem is shown that, if b is non-zero, then the shift with the vector b is an hypercyclic operator. Maclane in 1952 showed that the Differentiation operator on $H(C)$ is an hypercyclic operator. Bourdon and Shapiro also studied the behavior of composition operators on this space. For more information reader can see [1–14].

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1 Introduction

Let X be a frechet space and T be a bounded linear operator on X. For each $x \in X$ put

$$Orb(T, x) = \{T^n(x) : n \geq 0)\} = \{x, Tx, T^2x, T^3x, \ldots\}$$

The set $Orb(T, x)$ is called orbit of vector x under the operator T and the operator T is called hypercyclic operator if there exist vector x in X such that the set $Orb(T, x)$ is dense in X, that is

$$\overline{Orb(T, x)} = \overline{\{x, Tx, T^2x, T^3x, \ldots\}} = X.$$

In this case a vector x is called hypercyclic vector for the operator. If X^* be the dual of space X and both operators $T : X \to X$ and $T^* : X^* \to X^*$ are hypercyclic, then the operator T is called dual hypercyclic.(Note that if X be a bounded space, then the dual of X denoted by X'). If M is a subset of X and all non-zero element of space M be a hpercyclic vector for T, then M is called hypercyclic subspace of X.

2 Preliminary Notes

Suppose that $H(C)$ be the space of all functions of one complex variable with the uniform convergence topology on compact subsets of C. Consider Banach space E, Ferechet algebra by elements of dual space with uniform convergence topology over the balls of E. Space $H_{bc}(E)$ containing all bounded functions on compact subsets of E, the space $H_{bc}(E)$ includes all functions $f = \sum_{n=0}^{\infty} P_n$ in which

$$P_n \in \overline{Span\{\varphi^n : \varphi E^*\}} \quad , \quad n = 0, 1, 2, 3, ...$$

$$\|P_n\|^{\frac{1}{n}} = (Sup_{\|x\|<1}|P_n|)^{\frac{1}{n}} \to 0 \quad , \quad n \to \infty$$

The operator $\phi : H(E) \to H(E)$ by definition $\phi(f) = Df$ is called differentiation operator. Let $\varphi \in H(C)$, then the operator $C_\varphi : H(C) \to H(C)$ by definition $C_\varphi(f) = fo\varphi$ on $H(C)$ is hypercyclic, if and only if the operator φ is a shift with a non-zero vector $b \in C$. In other words, $\exists 0 \neq b \in C, \varphi(z) = z+b$(see[1]). Differentiation operator on $H(C)$ is a hypercyclic operator(see[6]).

If $\phi(z) = \sum_{|\alpha|\geq 0} C_\alpha z^\alpha$ be non-constant entire function on C, Then the operator $\phi_D : H(C^n) \to H(C^n)$ by definition $\phi_D(f) = \sum_{|\alpha|\geq 0} C_\alpha D^\alpha f, f \in H(C)$ is hypercyclic operator(see[10]). Also all continuous linear operator on $H(C^n)$ substitute with translation, if and only if, be for one $\varphi \in H(C^n)$ is of exponential form $T = \varphi' D$ (see[10]). Let X be an F-space and $T : X \to X$ be a continuous linear operator and assume that X_0, Y_0 are two dense subsets of X and $\{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers, and there are sequences $S_{n_k} : Y_0 \to X$ of mapping such that,
(1). $T^{n_k} \to 0, k \to \infty$, *Pointwise on* X_0
(2). $S_{n_k} \to 0, k \to \infty$, *Pointwise on* Y_0
(3). $T^{n_k} S_{n_k} = I_{Y_0}$
then the operator T is hypercyclic(Hypercyclic Creterion)

3 Main Results

Theorem 3.1 *The collection $B = \{e^\varphi : \varphi \in E^*\}$ is a independently linear subset of $H_{bc}(E)$.*

Proof. Let $\{e^{\varphi_i}\}_{i\in I}$ be the maximal independently linear subset of B. Suppose $\varphi \in E^*$ be fix point, and $\zeta_{i_1}, \zeta_{i_2}, ..., \zeta_{i_r} \in C$ with the condition,

$$\zeta_{i_1}.e^{\varphi_{i_1}} + \zeta_{i_2}.e^{\varphi_{i_2}} + \ldots + \zeta_{i_r}.e^{\varphi_{i_r}} = e^\varphi$$

Let α be a trivial element of E. Use the Operator $f \to df(.)\alpha$ in bellow equation, we have,

$$df(\zeta_{i_1}.e^{\varphi_{i_1}} + \zeta_{i_2}.e^{\varphi_{i_2}} + \ldots + \zeta_{i_r}.e^{\varphi_{i_r}})(\alpha) = df(e^\varphi)$$

so

$$df(\zeta_{i_1}.e^{\varphi_{i_1}})(\alpha) + df(\zeta_{i_2}.e^{\varphi_{i_2}})(\alpha) + \ldots + df(\zeta_{i_r}.e^{\varphi_{i_r}})(\alpha) = df(e^\varphi)(\alpha)$$
$$\zeta_{i_1}.\varphi_{i_1}(\alpha).e^{\varphi_{i_1}} + \zeta_{i_2}.\varphi_{i_2}(\alpha).e^{\varphi_{i_2}} + \ldots + \zeta_{i_r}.\varphi_{i_r}(\alpha).e^{\varphi_{i_r}} = \varphi(\alpha).e^\varphi$$

Since $\{e^{\varphi_i}\}_{i\in I}$ is a independently linear subset of B and $\zeta_{i_1}, \zeta_{i_2}, ..., \zeta_{i_r} \in C$ are non-zero elements, then

$$\varphi_{i_1}(\alpha) = \varphi_{i_2}(\alpha) = \ldots = \varphi_{i_r}(\alpha)$$

Now, since α be a trivial element of E, then

$$\varphi_{i_1} = \varphi_{i_2} = \ldots = \varphi_{i_r}$$

So $\{e^{\varphi_i}\}_{i\in I} = C^*$, by this the proof is complete.

Theorem 3.2 *Let U be an open subset of E^*, then $S = Span\{e^{\varphi} : \varphi \in U\}$ is a dense subset of $H_{bc}(E)$.*

Proof. Let $\varphi_0 \in E^*$ and $\Lambda : H_{bc}(E) \to H_{bc}(E)$ bye $\Lambda(\psi) = e^{\varphi_0}.\psi$. Suppose $\psi_1, \psi_2 \in H_{bc}(E)$ and $\Lambda(\psi)_1 = \Lambda(\psi)_2$, so $e^{\varphi_0}.\psi_1 = e^{\varphi_0}.\psi_2$. Since $e^{\varphi_0} \neq 0$, then $\psi_1 = \psi_2$, that is the operator Λ is one-one operator. Since constant operator and identity operator are continuous, then the operator Λ is continuous. Now since $\Lambda(\psi)^{-1} = e^{-\varphi_0}.\psi$ is continuous operator, then the operator Λ is a homeomorphism and

$$\overline{Span\{e^{\varphi_0+\varphi} : \varphi \in U\}} = H_{bc}(E) \Leftrightarrow \overline{Span\{e^{+\varphi} : \varphi \in U\}} = H_{bc}(E)$$

If $\lambda_o \in U$ then take $U_0 = \{\varphi - \lambda_0 : \varphi \in U\}$, then $0 = \lambda_0 - \lambda_0 \in U_0$, So without lost of generality we can suppose $0 \in U$. If U be a non-empty open subset of E^*, such that the norm of all element in U are not zero, then theorem is trivial. So assume that $\varphi_0 \in U, \|\varphi_0\| \neq 0$ and define

$$U_0 = \{\frac{1}{\|\varphi_0\|}\varphi : \varphi \in U\}$$

Now we have

$$\|\frac{1}{\varphi_0}\varphi_0\| = \|\frac{1}{\varphi_0}\|.\|\varphi_0\| = 1 \quad , \quad \frac{1}{\|\varphi_0\|}\varphi_0 \in U_0$$

So we have an open non-empty subset of E^* contain an element of norm 1. Now take $\delta > 0$ such that,

$$U = \{\varphi \in E^* : \|\varphi\| < \delta\}$$

Specially, for $0 \in U$ we have $1 \in \overline{U}$, Now we just to proof that,

$$\varphi^n \in \overline{S} \quad , \quad \forall n \geq 0 \quad , \quad \forall \varphi \in U$$

For this, suppose that $\varphi^n \in U$ for $\varphi^n \in U$ and $n \leq k-1$. In this way we have

$$\psi_t = \frac{e^{t_\varphi} - 1 - t\varphi - \frac{(t\varphi)^2}{2!} - \ldots - \frac{(t\varphi)^k}{k!}}{t^k}$$

Since $t\varphi \in U$, assume that $x \in E$ be given, then

$$|(\psi_t - \frac{\varphi^k}{k!})(x)| = |\frac{1}{t^k}(e^{t_\varphi} - 1 - t\varphi - \frac{(t\varphi)^2}{2!} - \ldots - \frac{(t\varphi)^k}{k!})(x)|$$

$$\leq t \sum_{n \geq k+1} t^{n-k-1} \frac{|\varphi(x)|^n}{n!} \leq te^{\delta\|x\|}$$

Then in the space $H_{bc}(E)$ we have

$$\psi_t \rightarrow \frac{\varphi^k}{k!} \quad , \quad t \rightarrow \infty$$

So $\frac{\varphi^k}{k!} \in \overline{S}$, and by this the proof is complete.

Theorem 3.3 *Let E is a banach space with dual E^* and $0 \neq \alpha \in E$ be Non-constant member in $H(C)$ of the exponential type $\phi(z) = \sum_{n=0}^{\infty} C_n z^n$, Then the operator $\phi_\alpha(D) : H_{bc}(E) \rightarrow H_{bc}(E)$ defined by $\phi_\alpha(D) = \sum_{n=0}^{\infty} c_n \overline{D^n f(.)}\alpha$ is a hypercyclic operator.*

Proof. Let $\psi : E_* \rightarrow C$ defined by $\psi(\varphi) = \sum_{n=0}^{\infty} P_n(\varphi)$, in which each function $P_n : E_* \rightarrow C$ be n-similar polynomials, such that for each $n \geq 0$ are defined by $P_n(\varphi) = c_n\varphi_n(\alpha)$. Now, since φ is the exponential type, so there is $R > 0$ so that we have $|c_n| \leq \frac{R_n}{n!}$ for each $n \geq 0$. Consider $\varphi \in E_*$ with the condition $\|\varphi\| \leq 1$, Now for $n \geq 0$, we have

$$|P_n(\varphi)| = |c_n\varphi_n(\alpha)| = |c_n|.|\varphi_n(\alpha)| \leq \frac{R_n}{n!}\|\varphi\|^n.\|\alpha\|^n \leq (\frac{c.\|\alpha\|.R}{n})^n$$

$$|P_n(\varphi)| \leq (\frac{c.\|\alpha\|.R}{n})^n$$

$$|P_n(\varphi)|_{\frac{1}{n}} \leq (\frac{c.\|\alpha\|.R}{n})$$

then

$$|P_n(\varphi)|^{\frac{1}{n}} \rightarrow 0 \quad , \quad n \rightarrow \infty$$

so $\psi : E^* \rightarrow C$ is entire, bounded and nonconstant. Since $\psi : E^* \rightarrow C$ is nonconstant, then the sets

$$U = \{\varphi \in E^* : |\sum_{n=0}^{\infty} P_n(\varphi)| = |\psi(\varphi)| < 1\}$$

and

$$V = \{\varphi \in E^* : |\sum_{n=0}^{\infty} P_n(\varphi)| = |\psi(\varphi)| > 1\}$$

are open and non-empty, so by theorem2.2 two sets

$$X_0 = Span\{e^\varphi : \varphi \in U\} \quad , \quad Y_0 = Span\{e^\varphi : \varphi \in V\}$$

are dense subspaces of $H_{bc}(E)$. If $T = \Phi(D)$ then

$$T(e^{\varphi}) = \sum_{0}^{\infty} c_n \overline{D^n(e^{\varphi})}\alpha = \sum_{0}^{\infty} c_n \varphi^n(\alpha) e^{\varphi} = e^{\varphi} \sum_{0}^{\infty} c_n \varphi^n(\alpha) = e^{\varphi}\psi(\varphi)$$

by theorem2.

$$T^n \to 0 \quad , \quad n \to \infty \quad , \quad pointwise\ on\ X_0$$

Also by theorem2.1 we conclude that, there is a map $S : Y_0 \to Y_0$ defined by $S(e^{\varphi}) = |\psi(\varphi)|e^{\varphi}$ such that,

$$S^n \to 0 \quad , \quad n \to \infty \quad , \quad pointwise\ on\ Y_0$$

and $TS = I_Y$
so by Kitai-Getner-Shapiro theorem the operator $T = \Phi_\alpha(D)$ is hypercyclic operator.

Corollary 3.4 *If E be a Banach space with separable dual E^*, then the space $H_{bc}(E)$ admitted a hypercyclic operator.(Also E^*)*

Corollary 3.5 *Let E be a Banach space with separable dual E^* and $0 \neq \alpha \in E$, then operator T_α definition by $T_\alpha f(x) = f(x+\alpha)$ on $H_{bc}(E)$ is a hypercyclic operator.*

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