Int. J. Contemp. Math. Sciences, Vol. 6, 2011, no. 34, 1681 - 1689

Binormal and Idempotent Integral Operators

Anupama Gupta

Govt.Collage for Women Parade, Jammu, J & K, India
anu_ju08@yahoo.com

B. S. Komal

Department of Engineering Mathematics
M.I.E.T, Kot Bhalwal J & K, India
bskomal 2 @Yahoo.co.in

Abstract

In this paper we study the kernel of integral operators as well as kernel of composite integral operator. The conditions for composite integral operator to be binormal, idempotent and projection are obtained.

Mathematics Subject Classification: Primary 47B20, Secondary 47B38

Keywords: Integral operator, composite integral operator, separable kernel, binormal operator, idempotent operator, projection operator

1 Preliminaries

Let (X, S, μ) is a σ-finite measure space and let $\phi : X \to X$ be a non- singular measurable transformation ($\mu(E) = 0 \Rightarrow \mu\phi^{-1}(E) = 0$). Then a composition transformation, for $1 \leq p < \infty, C_\phi : L^p(\mu) \to L^p(\mu)$ is defined by $C_\phi f = fo\phi$ for every $f \in L^p(\mu)$. In case C_ϕ is continuous, we call it a composition operator induced by ϕ. It is easy to see that C_ϕ is a bounded operator if and only if $\frac{d\mu\phi^{-1}}{d\mu} = f_o$, the Radon-Nikodym derivative of the measure $\mu\phi^{-1}$ with respect to the measure μ, is essentially bounded. For more detail about composition operators we refer to Nordgen [5], Shapiro [6], Singh and Manhas [10], Singh and Komal [9].

A kernel $K \in L^p(\mu \times \mu)$ always induces a bounded integral operator

$$T_K : L^p(\mu) \to L^p(\mu)$$

defined by

$$(T_K f)(x) = \int K(x,y) f(y) d\mu(y).$$

Given a kernel K and a non-singular measurable function $\phi : X \to X$, the composite integral operator T_{K_ϕ} induced by (K, ϕ) is a bounded linear operator $T_{K_\phi} : L^p(\mu) \to L^p(\mu)$ defined by

$$\begin{aligned} (T_{K_\phi} f)(x) &= \int K(x,y) f(\phi(y)) d\mu(y) \\ &= \int K_\phi(x,y) f(y) d\mu(y) \end{aligned}$$

Set

$$\begin{aligned} K_\phi(x,y) &= (E(K(x,.))o\phi^{-1} f_o)(y) \\ &= E(K(x, \phi^{-1}(y)) f_o(y)) \\ &= E(K_x(\phi^{-1}(y)) f_o(y)) \end{aligned}$$

where E is called conditional expectation. For more properties of the expectation operator, see Parthasarthy [7]and Lambert [4]. The integral operators received considerable attention of several mathematicians like Halmos and Sunder [8], Bloom and Kerman [12], Gupta and Komal ([1],[2],[3]). In his paper Whitley [11] established the Lyubic,s conjecture [13] and generalized it to Volterra composition operator on $L^p[0,1]$.Gupta and Komal [1] established the conditions which characterize the Hermitian and normal composite integral operators. In this paper we study the some basic operator theoretic properties of composite integral operators. Some conditions which characterize binormal projection and idempotent composite integral operators are established.
A is binormal if $A^*AAA^* = AA^*A^*A$

Theorem 1.1 *Let T_K be an integral operator induced by separable kernel on $L^2(\mu)$. Then*

$$Ker\ T_K = span(\{b_1, b_2, ..., b_n\})^\perp$$

and

$$Ker\ T_K^* = span(\{a_1, a_2, ..., a_n\})^\perp.$$

Proof: *Firstly, suppose $f \in span(\{b_1, b_2, ...b_n\})^\perp$ and*

$$K(x,y) = \sum_{i=1}^{n} a_i(x) b_i(y).$$

Then

$$\begin{aligned}(T_K f)(x) &= \int K(x,y)f(y)d\mu(y)\\ &= \int [a_1(x)b_1(y) + a_1(x)b_2(y) + \ldots.. + a_n(x)b_n(y)]f(y)d\mu(y).\\ &= a_1(x)\int b_1(y)f(y)d\mu(y) + a_2(x)\int b_2(y)f(y)d\mu(y)\\ &\qquad + \ldots\ldots + a_n(x)\int b_n(y)f(y)d\mu(y) = 0.\end{aligned}$$

Therefore $f \in KerT_K$. *Thus span* $(\{b_1, b_2, \ldots..b_n\})^{\perp} \subset KerT_K$.
Conversely, suppose $f \in KerT_K$.
Then$(T_K f(x) = 0$ *for* μ *- almost all* $x \in X$.

$$\Rightarrow \quad a_1(x)\int b_1(y)f(y)d\mu(y) + a_2\int b_2(y)f(y)d\mu(y) + \ldots.. + a_n\int b_n(y)f(y)d\mu(y) = 0$$

$$\Rightarrow \quad \int_x b_1(y)f(y)d\mu(y) = 0, \int_x b_2(y)f(y)d\mu(y) = 0, \ldots.. \int b_n(y)f(y)d\mu(y) = 0,$$

as $a_1(x), a_2(x), \ldots..a_n(x)$ *are linearly independent.*
Hence $f \in span(\{b_1, b_2, \ldots\ldots, b_n\})^{\perp}$.
This proves that

$$Kerf = span(\{b_1, b_2, \ldots\ldots, b_n\})^{\perp}.$$

Next, suppose $g \in span(\{a_1, a_2, \ldots.., a_n\})^{\perp}$.
Then

$$\begin{aligned}\langle T_K^* g, f\rangle &= \langle g, T_K f\rangle\\ &= \int g(x)(T_K f)(x)d\mu(x)\\ &= \int g(x)\int K(x,y)f(y)d\mu(y)d\mu(x)\\ &= \int g(x)[\int \sum a_i(x)b_i(y)f(y)d\mu(y)]d\mu(x),\\ &= \int_X [\sum_{i=1}^{n}\int_X a_i(x)b_i(y)g(x)f(y)d\mu(y)]d\mu(x),\\ &= \sum_{i=1}^{n}\int_X [\int_X a_i(x)b_i(y)g(x)f(y)d\mu(y)]d\mu(x),\\ &= \sum_{i=1}^{n}\int_X [\int_X a_i(x)g(x)d\mu(x)]b_i(y)f(y)d\mu(y),\\ &= \quad 0 \textit{ for every } f \in L^2(\mu).\end{aligned}$$

Therefore, $span(\{a_1, a_2, ...a_n\})^{\perp} \subset KerT_K^*$.
To prove the inclusion other way, let $T_K^* f = 0$.
Then $(T_K^* f)(x) = 0$ *for* μ*-almost all x.*
or $\int K^*(x, y)f(y)d\mu(y) = 0$ *for* μ *almost all x.*
or $\int \bar{K}(y, x)f(y)d\mu(y) = 0$ *for* μ *almost all x.*
or $\int \sum_{i=1}^{n} a_i(y)b_i(x)f(y)d\mu(y) = 0$ *for* μ *almost all x.*
or $\sum_{i=1}^{n} b_i(x)(\int a_i(y)f(y)d\mu(y)) = 0.$
$\Rightarrow$ $\int a_i(y)f(y)d\mu(y) = 0$ *for each i = 1, 2,....n*
$\Rightarrow$ $f \perp a_i$ *for each i = 1, 2,....., n .*
$\Rightarrow$ $f \in span(\{a_1, a_2, ..., a_n\})^{\perp}$.
Thus $KerT_K^* \subset span(\{a_1, a_2, ..., a_n\})^{\perp}$,
so that

$$KerT_K^* = span(\{a_1, a_2, ..., a_n\})^{\perp}$$

Corollary 1.2 *Let* (X, S, μ) *be a non-atomic measure space. Suppose K is a separable kernel. Then* T_K *is never injective on* $L^2(X, S, \mu)$.
Proof: *Suppose* $K(x, y) = \sum_{i=1}^{n} a_i(x).b_i(y)$ *where* $a_1(x), ..., a_n(x)$*, and* $b_1(y), b_2(y)b_n(y)$ *are linearly independent vectors. Then* $span\{a_1(x), a_2(x), ..., a_n(x)\}$ *is finite dimensional.*
Choose a non-zero vector $f \in span(\{a_1(x), a_2(x), ...a_n(x)\})^{\perp}$.
Then $T_K f = 0$*, so that* T_K *has non-trivial kernel.*

Corollary 1.3 *Let* (X, S, μ) *be a finite atomic measure space and K be a separable kernel. Then* T_K *is injective if and only if X has n atoms.*
Proof: *Suppose X has n atoms. Then* $L^2(X, S, \mu)$ *is n-dimensional space. Therefore*

$$\begin{aligned} KerT_K &= span(\{a_1(x), a_2(x), ..., a_n(x)\})^{\perp} \\ &= (L^2(X, S, \mu))^{\perp} \\ &= \{0\} \end{aligned}$$

Theorem 1.4 *Suppose K(x, y) is a separable kernel. Then*

$$KerT_{K_\phi} = span(\{f_o E(b_1) o\phi^{-1}, f_o E(b_2) o\phi^{-1}, ...f o E(b_n) o\phi^{-}1\})^{\perp}$$

Proof: *Suppose* $f \perp span\{f_oE(b_1)o\phi^{-1}, f_oE(b_2)o\phi^{-1}, ..., f_oE(b_n)o\phi^{-1}\}$. *Then*

$$\begin{aligned}(T_{K_\phi}f)(x) &= \int_X K(x,y)f(\phi(y))d\mu(y) \\ &= \int_X \sum_{i=1}^{n} a_i(x)b_i(y)f(\phi(y))d\mu(y) \\ &= \sum_{i=1}^{n} a_i(x)\int_X b_i(y)f(\phi(y))d\mu(y) \\ &= \sum a_i(x)\int f_oE(b_i)o\phi^{-1}f(y)d\mu(y) \\ &= 0\end{aligned}$$

for μ*- almost all* $x \in X$.
Hence

$$Span(\{f_oE(b_1)o\phi^{-1}, f_oE(b_2)o\phi^{-1}, ..., f_oE(b_n)o\phi^{-1}\})^\perp \subset KerT_{K_\phi}.$$

Again if $f \in KerT_{K_\phi}$, *then*

$$\sum_{i=1}^{n} a_i(x)\int f_o(y)E(b_i)o\phi^{-1}f(y)d\mu(y) = 0$$

$\Rightarrow \quad \int_X f_o(y)E(b_i)o\phi^{-1}f(y)d\mu(y) = 0$
$\Rightarrow \quad f \perp f_o.E(b_io\phi^{-1})$ *for* $i = 1, 2, ..., n$.
or $\quad f \perp span(\{f_o.E(b_1)o\phi^{-1}, f_oE(b_2)o\phi^{-1}, ..., f_oE(b_n)o\phi^{-1}\})$.
This proves that

$$KerT_{K_\phi} \subseteq span(\{f_o.E(b_1)o\phi^{-1}, f_oE(b_2)o\phi^{-1}, ..., f_oE(b_n)o\phi^{-1}\})^\perp$$

Thus

$$KerT_{K_\phi} = span(\{f_o.E(b_1)o\phi^{-1}, f_oE(b_2)o\phi^{-1}, ..., f_oE(b_n)o\phi^{-1}\})^\perp$$

Theorem 1.5 *Let* $K_\phi \in L^2(\mu \times \mu)$. *Then* T_{K_ϕ} *is binormal if and only if*

$$\int\int\int K_\phi^*(x,y)K_\phi(y,z)K_\phi(z,t)K_\phi^*(t,p)d\mu(y)d\mu(z)d\mu(t)$$

$$= \int\int\int K_\phi(x,y)K_\phi^*(y,z)K_\phi^*(z,t)K_\phi(t,p)d\mu(y)d\mu(z)d\mu(t).$$

Proof: *Suppose the condition is true.*
For $f, g \in L^2(\mu)$, we have

$$\begin{aligned}
\langle T^*_{K_\phi} T_{K_\phi} T_{K_\phi} T^*_{K_\phi} f, g\rangle &= \int (T^*_{K_\phi} T_{K_\phi} T_{K_\phi} T^*_{K_\phi} f)(x)\bar{g}(x)d\mu(x) \\
&= \int\int [K^*_\phi(x,y)(T_{K_\phi} T_{K_\phi} T^*_{K_\phi} f)(y) \\
&\qquad d\mu(y)]\bar{g}(x)d\mu(x) \\
&= \int\int K^*_\phi(x,y)(\int K_\phi(y,z)(T_{K_\phi} T^*_{K_\phi} f)(z) \\
&\qquad d\mu(z))d\mu(y)\bar{g}(x)d\mu(x) \\
&= \int\int\int K^*_\phi(x,y)K_\phi(y,z)(\int K_\phi(z,t)(T^*_{K_\phi} f)(t) \\
&\qquad d\mu(t))d\mu(z)d\mu(y)\bar{g}(x)d\mu(x) \\
&= \int\int\int\int K^*_\phi(x,y)K_\phi(y,z)K_\phi(z,t)(\int K^*_\phi t,p)f(p) \\
&\qquad d\mu(p))d\mu(t)d\mu(z)d\mu(y)\bar{g}(x)d\mu(x) \\
&= \int\int\int\int\int K^*_\phi(x,y)K_\phi(y,z)K_\phi(z,t)K^*_\phi t,p)f(p) \\
&\qquad d\mu(p))d\mu(t)d\mu(z)d\mu(y)\bar{g}(x)d\mu(x) \\
&= \int\int\int\int\int K^*_\phi(x,y)K_\phi(y,z)K_\phi(z,t)K^*_\phi(t,p)f(p) \\
&\qquad d\mu(y)d\mu(z)d\mu(t)f(p)d\mu(p)\bar{g}(x)d\mu(x) \qquad ...(1)
\end{aligned}$$

and

$$\begin{aligned}
\langle T_{K_\phi} T^*_{K_\phi} T^*_{K_\phi} T_{K_\phi} f, g\rangle &= \int (T_{K_\phi} T^*_{K_\phi} T^*_{K_\phi} T_{K_\phi} f(x)\bar{g}(x)d\mu(x) \\
&= \int\int\int\int\int K_\phi(x,y)K^*_\phi(y,z)K^*_\phi(z,t)K_\phi(t,p)f(p) \\
&\qquad d\mu(p))d\mu(t)d\mu(z)d\mu(y)\bar{g}(x)d\mu(x) \\
&= \int\int\int\int\int K_\phi(x,y)K^*_\phi(y,z)K^*_\phi(z,t)K_\phi(t,p)f(p) \\
&\qquad d\mu(y)d\mu(z)d\mu(t)f(p)\bar{g}(x)d\mu(x) \qquad ...(2)
\end{aligned}$$

It follows from (1) and (2) that T_{K_ϕ} is binormal.
Conversely, suppose T_{K_ϕ} is binormal. Take $f = \chi_E$ and $g = \chi_F$, we see that from (1) and (2)

$$\int\int_E\int_F K^*_\phi(x,y)K_\phi(y,z)K_\phi(z,t)K^*_\phi(t,p)d\mu(y)d\mu(z)d\mu(t)$$

$$\int\int_E\int_F K_\phi(x,y)K^*_\phi(y,z)K^*_\phi(z,t)K_\phi(t,p)d\mu(y)d\mu(z)d\mu(t)$$

for all $E, F \in S \times S$. Hence the required condition holds.

Theorem 1.6 *Let $T_{K_\phi} \in B(L^2(\mu))$. Then T_{K_ϕ} is an idempotent if and only if K_ϕ is an idempotent, i.e. $K_\phi^2 = K_\phi$ a.e where*

$$K_{\phi^2}(x, y) = \int K_\phi(x, z) K_\phi(z, y) dz.$$

Proof: *First, suppose K_ϕ is idempotent. Let $f, g \in L^2(\mu)$.*

$$\begin{aligned} \langle T_{K_\phi}^2 f, g\rangle &= \langle T_{K_\phi} f, T_{K_\phi}^* g\rangle \\ &= \int T_{K_\phi} f)(x)(T_{K_\phi}^* g)(x) d\mu(x) \\ &= \int (\int_X K_\phi(x, y) f(y) d\mu(y). \overline{\int K_\phi^*(x, z) g(z) d\mu(z))} d\mu(z) \end{aligned}$$

$$\begin{aligned} &= \int (\int_X \int K_\phi(x, y) K_\phi(z, x) f(y) \bar{g}(z) d\mu(z) d\mu(y)) d\mu(x) \\ &= \int\int (\int K_\phi(z, x) K_\phi(x, y) d\mu(x)) f(y) \bar{g}(z) d\mu(y)) d\mu(z) \\ &= \int (\int K_\phi^2(z, y) f(y) d\mu(y)) \bar{g}(z) d\mu(z) \\ &= \int (T_{K_\phi} f)(z) \bar{g}(z) d(z) \\ &= \langle T_{K_\phi} f, g\rangle. \end{aligned}$$

Hence T_{K_ϕ} is an idempotent.
Conversely, suppose T_{K_ϕ} is an idempotent.
That is,

$$\langle T_{K_\phi}^2 f, g\rangle = \langle T_{K_\phi} f, g\rangle \quad \forall \quad f, g \in L^2(\mu)$$

$$\int\int\int K_\phi(z, x) K(x, y) d\mu(x)) f(y) \bar{g}(z) d\mu(y) d\mu(z)$$
$$= \int\int K_\phi(z, x) f(x) \bar{g}(z) d\mu(x) d\mu(z)$$

$\Rightarrow \quad \int\int K_\phi^2(z, y) f(y) d\mu(y) \bar{g}(z) d\mu(z)$

$$= \int\int K_\phi(z, y) f(y) \bar{g}(z) d\mu(y) d\mu(z)$$

Taking $f = \chi_E$ and $g = \chi_F$ for $E, F \in S$.

$$\int_{E \times F} K_\phi^2(z, y) d\mu(y) d\mu(z) = \int_{E \times F} K_\phi(z, y) d\mu(y) d\mu(z).$$

This proves $K_\phi^2 = K_\phi$, a.e.
Hence K_ϕ is an idempotent.

Theorem 1.7 *Let $T_{K_\phi} \in B(L^2(\mu))$. Then T_{K_ϕ} is projection if and only if K_ϕ is an idempotent and $K_\phi(x, y) = K_\phi\overline{(y, x)}$.*
Proof: *The proof follows from the theorem 6.*

References

[1] A. Gupta and B.S. Komal, Composite integral operator on $L^2(\mu)$, Pitman Lecture Notes in Mathematics series 377, (1997), 92-99.

[2] A. Gupta and B.S. Komal, Generalize integral operaators on function spaces, Int. Journal of Math Analysis, Vol 3, 26(2009), 1277-1282.

[3] A. Gupta and B.S. Komal, Weighted composite integral operators, Int. Journal of Math Analysis, Vol 3, 26(2009), 1283-1293.

[4] A. Lambert, Normal extension of subnormal composition operators, Michigan Math. J., 35, (1988),443-450.

[5] E.A. Nordgen, Composition operators on Hilbert spaces, Lecture notes in Math, 693, Springer -Verlag, New York, (1978), 37-63.

[6] J.H. Shapiro, Composition operators and classical function theory, Springer- Verley, New York. (1993).

[7] K.R. Parthasarathy, Introduction to probability and measure , Macmillion Limited, (1977).

[8] P.R. Halmos and V.S. Sunder, Bounded integral operators on L^2-spaces, Springer-Verlag, New York, (1978).

[9] R.K. Singh and B.S. Komal, Composite integral operator on l^p and its adjoint, Proc. Amer. Math. Soc. 70, (1978), 21-25.

[10] R.K. Singh and J.S. Manhas, Composition operators on function spaces, North Holland Mathematics studies 179, Elsevier sciences publishers Amsterdam, New York (1993).

[11] R.Whitley, The spectrum of a Volterra composition operator, Integral equation and operator theory Vol. 10 (1997).

[12] S. Bloom and R. Kerman, Weighted norm inequalities for operators of Hardy type, Proc. Amer. Math. Soc. 113, (1991),135-141.

[13] Yu. I. Lyubic, Composition of integration and substitution, Linear and complex Analysis, Problem Book, Springer Lect. Notes in Maths., 1043, Berlin (1984), 249-250.

Received: March, 2011