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An Analytical Study for Two-Dimensional Schrodinger Equations by Using Homotopy Perturbation Method

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Abstract

In this work, an analytical approximation is introduced to obtain the exact solutions of the two-dimensional Schrodinger equations. The main objective is to propose alternative methods of solution, which do not require small parameters and avoid linearization and physically unrealistic assumptions. To illustrate the ability and reliability of the method some examples are provided. The results show that these methods are very efficient and convenient and can be applied to a large class of problems.

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1 Introduction

The solution of the two-dimensional Schrodinger equation has been a subject of considerable interest. This equation is the fundamental equation of physics for describing quantum mechanical behavior [1]. It is also often called

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the Schrodinger wave equation, and is a partial differential equation that describes how the wave function of a physical system evolves over time. Also this equation appears in electromagnetic wave propagations [2], in underwater acoustics (paraxial approximation of the wave equations [3]) or also in optic (Fresnel equation [4]) and design of certain optoelectronic devices [5] as it models an electromagnetic wave equation in a two-dimensional weakly guiding structure.
Consider the following two-dimensional Schrodinger equation with the following initial condition

$$-i\frac{\partial u}{\partial t}(x,y,t)=\frac{\partial^2 u}{\partial x^2}(x,y,t)+\frac{\partial^2 u}{\partial y^2}(x,y,t)+\omega(x,y)\,u(x,y,t), \qquad (1)$$

$$u(x,y,0)=\varphi(x,y), \qquad (x,y,t)\in[a,b]\times[a,b]\times[0,T],$$

Where $u(x,y,t)$ is the wave function in continuous and $\omega(x,t)$ is an arbitrary potential function and $i=\sqrt{-1}$.
In this letter, we will use homotopy perturbation method (HPM) to study two-dimensional Schrodinger equations. The homotopy perturbation method was first proposed by He [6] and was successfully applied to various engineering problems. In this method the solution is considered as the summation of an infinite series which usually converges rapidly to the exact solutions. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$which is considered as a small parameter. Considerable research works have been conducted recently in applying this method to a class of linear and non-linear equations.

2 Basic ideas of homotopy perturbation method

To explain this method, let us consider the following function:

$$A(u)-f(r)=0, \quad r\in\Omega, \qquad (2)$$

With the following boundary conditions:

$$B\left(u,{}^{\partial u}\!/_{\partial n}\right)=0, \ \ r\in\Gamma. \qquad (3)$$

Where A is a functional operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω. The operator A can generally speaking, be decomposed into two parts L and N, where L is a linear, and N is a non-linear operator, therefore, Eq.(2) can be rewritten as the following:

$$L(u)+N(u)-f(r)=0. \qquad (4)$$

We construct a homotopy $v(r,p) : \Omega \times [0,1] \to R$, which satisfies

$$H(v,p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[A(v) - f(r)\right] = 0, \quad r \in \Omega, \tag{5}$$

Or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - f(r)\right] = 0, \quad r \in \Omega, \tag{6}$$

Where $p \in [0,1]$ an embedding parameter, and u_0 is the first approximation that satisfies the boundary condition. Clearly, we have

$$H(v,0) = L(v) - L(u_0) = 0, \tag{7}$$

$$H(v,1) = A(v) - f(r) = 0, \tag{8}$$

The changing process of p from zero to unity is just that of $v(r,p)$ changing from $u_0(r)$ to $u(r)$. This is called deformation, and also, $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic in topology. If the embedding parameter$p$; $(0 \leq p \leq 1)$ is considered as a small parameter. Applying the classical perturbation technique, we can naturally that the solution of Eqs.(7) and (8) can be given as a power series in p, i.e.

$$v = v_0 + pv_1 + p^2 v_2 + \cdots, \tag{9}$$

And setting $p = 1$ results in the approximate solution of Eq.(2) as;

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 \cdots, \tag{10}$$

The above convergence is discussed in [6].

3 The homotopy perturbation method applied to two-dimensional Schrodinger equations

For solving Eq.(1), by homotopy perturbation method, we construct the following homotopy:

$$(1-p)\left(\frac{\partial U}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial U}{\partial t} - i\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \omega(x,y)U\right)\right) = 0,$$

Or

$$\frac{\partial U}{\partial t} - \frac{\partial u_0}{\partial t} + p\left(-i\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \omega(x,y)U\right) + \frac{\partial u_0}{\partial t}\right) = 0, \tag{11}$$

Suppose that the solution of Eq. (11) to be in the following form

$$U = U_0 + PU_1 + P^2 U_2 + \ldots \tag{12}$$

Substituting Eq.(12) into Eq.(11), and equating the coefficients of the terms with the identical powers of p,

$$\begin{array}{lll}
p^0: & \frac{\partial U_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, & \\
p^1: & \frac{\partial U_1}{\partial t} + \frac{\partial u_0}{\partial t} - i\left(\frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 U_0}{\partial y^2} + \omega(x,y)\,U_0\right) = 0, & U_1(x,y,0) = 0 \\
p^2: & \frac{\partial U_2}{\partial t} - i\left(\frac{\partial^2 U_1}{\partial x^2} + \frac{\partial^2 U_1}{\partial y^2} + \omega(x,y)\,U_1\right) = 0, & U_2(x,y,0) = 0 \\
p^3: & \frac{\partial U_3}{\partial t} - i\left(\frac{\partial^2 U_2}{\partial x^2} + \frac{\partial^2 U_2}{\partial y^2} + \omega(x,y)\,U_2\right) = 0, & U_3(x,y,0) = 0 \\
\vdots & & \\
p^j: & \frac{\partial U_j}{\partial t} - i\left(\frac{\partial^2 U_{j-1}}{\partial x^2} + \frac{\partial^2 U_{j-1}}{\partial y^2} + \omega(x,y)\,U_{j-1}\right) = 0, & U_j(x,y,0) = 0 \\
\vdots & &
\end{array}$$

For simplicity we take

$$U_0(x,y,t) = u_0(x,y,t) = u(x,y,0) = \varphi(x,y), \tag{13}$$

Having this assumption we get the following iterative equation

$$U_j = i\int_0^t \left(\frac{\partial^2 U_{j-1}}{\partial x^2} + \frac{\partial^2 U_{j-1}}{\partial y^2} + \omega(x,y)\,U_{j-1}\right) dt, \qquad j = 1,2,3,... \tag{14}$$

Therefore, the approximated solutions of Eq.(1) can be obtained, by setting $p = 1$.

$$u = \lim_{p\to 1} U = U_0 + U_1 + U_2 + U_3 + ... \tag{15}$$

4 Illustrative examples

In this section, to illustrate and show the ability of the method we present results of the method on several test problems.

Example 1. We consider Eq.(1) with $a = 0$, $b = 1$, $\omega(x,y) = 0$ and the following initial condition, $\varphi(x,y) = (\sin(x) + \sin(y))$. The exact solution is given with

$$u(x,y,t) = e^{-it}(\sin(x) + \sin(y)). \tag{16}$$

He's homotopy perturbation method consists of the following scheme

$$\frac{\partial U}{\partial t} - \frac{\partial u_0}{\partial t} + p\left(-i\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + \frac{\partial u_0}{\partial t}\right) = 0.$$

Starting with $U_0(x,y,t) = u_0(x,y,t) = u(x,y,0) = \sin(x) + \sin(y)$, by using Eq.(14), we obtain the recurrence relation

$$U_j = i\int_0^t \left(\frac{\partial^2 U_{j-1}}{\partial x^2} + \frac{\partial^2 U_{j-1}}{\partial y^2}\right) dt, \qquad j = 1,2,3,...$$

And exact solution will be as

$$u(x,y,t) = \lim_{p\to 1} U(x,y,t) = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} (\sin(x) + \sin(y)) = e^{-it} (\sin(x) + \sin(y)).$$

Example 2. We consider Eq.(1) with $a = 0,\ b = 1,\ \omega(x,y) = 3 - 2\tanh^2(x) - 2\tanh^2(y)$ and the following initial condition, $\varphi(x,y) = \frac{i}{\cosh(x)\cosh(y)}$. The exact solution is given with

$$u(x,y,t) = \frac{ie^{it}}{\cosh(x)\cosh(y)}. \tag{17}$$

He's homotopy perturbation method consists of the following scheme

$$\frac{\partial U}{\partial t} - \frac{\partial u_0}{\partial t} + p\left(-i\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \left(3 - 2\tanh^2(x) - 2\tanh^2(y)\right)U\right) + \frac{\partial u_0}{\partial t}\right) = 0.$$

Starting with $U_0(x,y,t) = u_0(x,y,t) = u(x,y,0) = \frac{i}{\cosh(x)\cosh(y)}$, by using Eq.(14), we obtain the recurrence relation

$$U_j = i\int_0^t \left(\frac{\partial^2 U_{j-1}}{\partial x^2} + \frac{\partial^2 U_{j-1}}{\partial y^2} + \left(3 - 2\tanh^2(x) - 2\tanh^2(y)\right)U_{j-1}\right) dt,$$

j=1,2,3,... . And exact solution will be as

$$u(x,y,t) = \lim_{p\to 1} U(x,y,t) = \sum_{n=0}^{\infty} \frac{\frac{i(it)^n}{n!}}{\cosh(x)\cosh(y)} = \frac{ie^{it}}{\cosh(x)\cosh(y)}.$$

Example 3. We consider Eq.(1) with
$a = 0,\ b = 1,\ \omega(x,y) = -\frac{4x^2+4y^2-4x-4y+\beta^2-4\beta+2}{\beta^2}$ and the following initial condition, $\varphi(x,y) = e^{\left(-\frac{(x-0.5)^2}{\beta} - \frac{(y-0.5)^2}{\beta}\right)}$.
The exact solution is given with

$$u(x,y,t) = e^{\left(-\frac{(x-0.5)^2}{\beta} - \frac{(y-0.5)^2}{\beta} - it\right)}. \tag{18}$$

He's homotopy perturbation method consists of the following scheme

$$\frac{\partial U}{\partial t} - \frac{\partial u_0}{\partial t} + p\left(-i\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \left(-\frac{4x^2+4y^2-4x-4y+\beta^2-4\beta+2}{\beta^2}\right)U\right) + \frac{\partial u_0}{\partial t}\right) = 0.$$

Starting with $U_0(x,y,t) = u_0(x,y,t) = u(x,y,0) = e^{\left(-\frac{(x-0.5)^2}{\beta}-\frac{(y-0.5)^2}{\beta}\right)}$,by using Eq.(14), we obtain the recurrence relation

$$U_j = i\int_0^t \left(\frac{\partial^2 U_{j-1}}{\partial x^2} + \frac{\partial^2 U_{j-1}}{\partial y^2} + \left(-\frac{4x^2+4y^2-4x-4y+\beta^2-4\beta+2}{\beta^2}\right)U_{j-1}\right)dt,$$

j=1,2,3,... . And exact solution will be as

$$u(x,y,t) = \lim_{p\to 1} U(x,y,t) = \sum_{n=0}^{\infty}\frac{(-it)^n}{n!}e^{\left(-\frac{(x-0.5)^2}{\beta}-\frac{(y-0.5)^2}{\beta}\right)} = e^{-it}e^{\left(-\frac{(x-0.5)^2}{\beta}-\frac{(y-0.5)^2}{\beta}\right)}.$$

5 Conclusion

In this paper, we have successfully developed homotopy perturbation method to obtain the exact solutions of two-dimensional Schrodinger equation. It is apparently seen that these method are very powerful and efficient for solving different kinds of problems arising in various fields of science and engineering and present a rapid convergence for the solutions. Mohebbi and Dehghan in [7] reported the computed error for Examples 1-3, but by He's homotopy perturbation we obtain the exact solution.

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