

# Inverse Limits of Hochschild-Mostow Inverse Systems

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## Abstract

In this paper, we investigate inverse limits of Hochschild-Mostow inverse systems. For example, we show that if  $(G_i) \xrightarrow{g_i} (H_i)$  is a mapping of inverse systems of groups, where all the  $g_i$  are surjective and  $(Ker g_i)$  is a G-compact inverse system, then the limiting map  $\varprojlim g_i$  is surjective. Also we prove that if  $(V_i, \pi_{ji})$  is a Hochschild-Mostow inverse system and  $V = \varprojlim V_i$ , then each projection  $\pi_i : V \rightarrow V_i$  is a closed map.

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## 1 Introduction

This paper starts with a clear concise introduction to some of the basic facts needed from the theory of inverse limits that includes definitions, examples, and some basic theorems of the theory of inverse limits. Such an introduction is given to provide a convenient repository for all readers. Other introductions can be found in [1], [2], [3], [4], [5], [6]. In Section 3, we investigate the inverse limit of a Hochschild-Mostow inverse system, i.e.  $\varprojlim X_i$ , where each  $X_i$  is a non-empty set that can be equipped with a compact  $T_1$  topology such that each transition map is a closed continuous map. Section 4 contains new results concerning inverse limits of Hochschild-Mostow inverse systems. For example, we prove that if  $(V_i, \pi_{ji})$  is a Hochschild-Mostow inverse system and  $V = \varprojlim V_i$  (with its inverse limit topology), then each projection  $\pi_i : V \rightarrow V_i$  is a closed map. We also prove that if  $(V_i, \pi_{ji})$  is a Hochschild-Mostow inverse system, then,  $\varprojlim V_i$  (with its inverse limit topology) is compact. This theorem was proved by Stone in [7], but here we give a new proof.

## 2 Preliminary Notes

**Definition 2.1** Let  $I$  be a set with a partial ordering  $\leq$ . Suppose  $I$  is directed upwards, i.e., for every  $i, j \in I$  there exists  $k \in I$  such that  $i \leq k$  and  $j \leq k$ . Let  $S = \{S_i : i \in I\}$  be a family of sets such that for every pair  $(i, j) \in I \times I$  with  $j \geq i$ , there is a map  $\pi_{ji} : S_j \rightarrow S_i$  satisfying the following two conditions:

1.  $\pi_{ii}$  is the identity map for every  $i \in I$ ;
2. if  $i \leq j \leq k$ , then  $\pi_{ki} = \pi_{ji} \circ \pi_{kj}$ .

Let  $\pi = \{\pi_{ji}, i, j \in I, j \geq i : \pi_{ji} : S_j \rightarrow S_i\}$

Then  $(S, \pi)$  or  $(S_i, \pi_{ji})_{i, j \in I}$  or simply  $(S_i)$  is called an inverse system and the maps  $\pi_{ji} : S_j \rightarrow S_i$  are called the transition maps of the inverse system.

The inverse limit of this system, denoted by  $\varprojlim S_i$ , is the subset of the Cartesian product  $\prod_{i \in I} S_i$  consisting of all elements  $s = (s_i)_{i \in I}$  such that  $\pi_{ji}(s_j) = s_i$  for every  $j \geq i$ . In other words,  $\varprojlim S_i$  is the set of all families of elements  $(s_i)_{i \in I}$  which are compatible with the transition maps  $\pi_{ji} : S_j \rightarrow S_i$ .

**Remark 2.2** If  $(S_i)$  is an inverse system, let  $\pi_i : \varprojlim S_i \rightarrow S_i$  be the canonical projection sending  $(s_i)$  to  $s_i$ . Then  $\pi_i = \pi_{ji} \circ \pi_j$  for every  $j \geq i$ .

**Remark 2.3** A surjective inverse system is an inverse system whose transition maps are surjective.

**Remark 2.4** An inverse sequence is an inverse system whose index set is  $I = \mathbb{N}$  (directed by its natural order).

**Note 2.5** In general there are two fundamental problems:

If  $(S_i)$  is an inverse system of non-empty sets, then

1.  $\varprojlim S_i$  may be empty
2. If each transition map is surjective, then the canonical projection  $\pi_i : \varprojlim S_i \rightarrow S_i$  may fail to be surjective. i.e, each  $s_i$  in  $S_i$  may fail to be part of a compatible family  $(s_i)$  in  $\varprojlim S_i$ .

**Definition 2.6** A collection of sets  $(A_i)$  is called a subinverse system of an inverse system  $(G_i)$ , and we write  $(A_i) \subset (G_i)$  if  $A_i \subset G_i$  for all  $i \in I$  and  $\pi_{ji}(A_j) \subseteq A_i$  for all  $j \geq i$ .

**Example 2.7** Let  $(S, \pi)$  be an inverse sequence where  $S_n = \mathbb{N}$  for every integer  $n$ . If  $\pi_{n+1, n}(x) = x + 1$ , then  $\varprojlim S_n = \emptyset$ .

*Proof.* By contradiction, suppose that  $\varprojlim S_n \neq \emptyset$  and let  $(s)$  be an element of  $\varprojlim S_n$ . Then,  $(s) = (s_0, s_1, s_2, \dots)$ , where  $s_1 = s_0 - 1$ ,  $s_2 = s_0 - 2$ , and so on. Thus  $s_{s_0} = s_0 - s_0 = 0$ , and  $s_{s_0+1} = -1 \notin \mathbb{N}$ .

Thus,  $\varprojlim S_n$  is empty.

**Example 2.8** *Directed unions yield inverse systems:*

Let  $A = \cup A_i$  be a directed union of sets (directed by inclusion). That is, for each  $A_i, A_j$ , there exists  $A_k$  such that  $A_i \subset A_k$  and  $A_j \subset A_k$ . Let  $K$  be any fixed set and let  $\text{Fun}(A_i, K)$  be the set of all maps of  $A_i$  into  $K$ . If  $A_i \subset A_j$ , let  $\pi_{ji} : \text{Fun}(A_j, K) \rightarrow \text{Fun}(A_i, K)$  be the restriction map, and let

$$F = \{\text{Fun}(A_i, K), i \in I\} \text{ and } \pi = \{\pi_{j,i}, i, j \in I, j \geq i (\pi_{j,i} : \text{Fun}(A_j, K) \rightarrow \text{Fun}(A_i, K))\}.$$

Then  $(F, \pi)$  is an inverse system. Moreover, its inverse limit is bijective to  $\text{Fun}(A, K)$ .

*Proof.*  $(F, \pi)$  is clearly an inverse system. Moreover, define

$h : \text{Fun}(A, K) \rightarrow \varprojlim \text{Fun}(A_i, K)$  by  $h(f) = (f_i)$  where  $f_i$  is the restriction of  $f$  to  $A_i$ , then  $h$  is one to one.  $h$  is also onto : let  $(f_i) \in \varprojlim \text{Fun}(A_i, K)$ , so  $\pi_{ji}(f_j) = f_i$  for all  $j \geq i$  ( $A_i \subset A_j$ ).

Define  $f : A \rightarrow K$  by  $f(a) = f_i(a)$  where  $a \in A_i$ .

Then  $f$  is well defined because  $A$  is directed by inclusion, and for all  $a \in A_i$ ,  $f(a) = f_i(a)$ . So  $f_i$  is the restriction of  $f$  to  $A_i$ , and  $h(f) = (f_i)$ .

**Example 2.9** (*Waterhouse*). *In the above example, let  $A$  be any uncountable directed set, and let  $\{A_i\}$  be the family of all finite subsets of  $A$ , then  $A = \cup A_i$ . Let  $K = \mathbb{N}$ , and let  $\text{Fun}(A_i, \mathbb{N})$  be the set of all 1-1 maps of  $A_i$  into  $\mathbb{N}$ . If  $A_i \subset A_j$ , let  $\pi_{ji} : \text{Fun}(A_j, \mathbb{N}) \rightarrow \text{Fun}(A_i, \mathbb{N})$  be the restriction map and let  $F = \{\text{Fun}(A_i, \mathbb{N}), i \in I\}$  and let  $\pi = \{\pi_{j,i}, i, j \in I, j \geq i (\pi_{j,i} : \text{Fun}(A_j, \mathbb{N}) \rightarrow \text{Fun}(A_i, \mathbb{N}))\}$ . Then, according to the previous example the inverse limit of the surjective inverse system  $(F, \pi)$  is bijective to the set of all 1-1 maps of  $A$  into  $\mathbb{N}$  which is empty since an uncountable set cannot be embedded in  $\mathbb{N}$ .*

**Example 2.10** *Let  $\mathbb{R}_n[x]$  be the set of all polynomials of degree at most  $n$  with coefficients in  $\mathbb{R}$ , and consider the inverse system*

$$\mathbb{R} \leftarrow \mathbb{R}_1[x] \leftarrow \mathbb{R}_2[x] \leftarrow \dots \text{with } \pi_{n,n-1}(a_0 + a_1x + \dots + a_nx^n) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}. \text{ Then } \varprojlim \mathbb{R}_n[x] = \mathbb{R}[[x]], \text{ the power series on } x.$$

**Definition 2.11** *Let  $(S_i, \pi_{ji})$  and  $(X_i, \pi'_{ji})$  be two inverse systems over the same index set  $I$ . Suppose that for every  $i \in I$  we are given maps  $f_i : S_i \rightarrow X_i$  such that  $\pi'_{ji} \circ f_j = f_i \circ \pi_{ji}$  for every  $i \leq j$ . That is, each such diagram*

$$\begin{array}{ccc} S_j & \xrightarrow{f_j} & X_j \\ \pi_{ji} \downarrow & & \pi'_{ji} \downarrow \\ S_i & \xrightarrow{f_i} & X_i \end{array}$$

commutes. Then there exists a unique map  $f: \varprojlim S_i \rightarrow \varprojlim X_i$  such that  $\pi'_i \circ f = f_i \circ \pi_i$  for every  $i \in I$ . We write  $f = \varprojlim f_i$ .

**Remark 2.12** Let  $f: \varprojlim G_i \rightarrow \varprojlim G'_i$  be a mapping of inverse systems of groups (whose transition maps are homomorphisms). Then  $\text{Ker } f = \varprojlim \text{Ker } f_i$ . This shows immediately that if we have a family of exact sequences of groups

$$1 \rightarrow (G_i) \xrightarrow{id} (G'_i) \xrightarrow{(f_i)} (G''_i) \rightarrow 1$$

which are compatible with all transition maps. Then the limiting sequence

$$1 \rightarrow (\varprojlim G_i) \xrightarrow{id} \varprojlim (G'_i) \xrightarrow{\varprojlim f_i} \varprojlim (G''_i) \rightarrow 1 \text{ is left exact.}$$

**Remark 2.13** Let  $(G, \pi)$  be an inverse system of groups where each  $\pi_{ji}$  is a group homomorphism. Then

1.  $\varprojlim G_i$  is a group
2. each projection map  $\pi_i: \varprojlim G_i \rightarrow G_i$  is a group homomorphism

A similar remark holds for rings, modules and vector spaces.

**Remark 2.14** Let  $(S, \pi)$  be a surjective inverse sequence. Then  $\varprojlim S_i \neq \phi$  and each projection  $\pi_i: \varprojlim S_i \rightarrow S_i$  is surjective.

**Remark 2.15** *Universal Property.* Let  $(X, \pi)$  be an inverse system of sets and let  $\pi_i: \varprojlim X_i \rightarrow X_i$  be the canonical projection. Then  $\varprojlim X_i$  has the following universal property. For every set  $Y$  with projections  $p_i: Y \rightarrow X_i$  such that  $\pi_{ji} \circ p_j = p_i$  for each  $i \leq j$ , there exists a unique map  $f: Y \rightarrow \varprojlim X_i$  such that  $\pi_i \circ f = p_i$  for every  $i \in I$ , where  $f$  is defined as follows  $f(y) = (p_i(y))$

**Definition 2.16** *Inverse limit topology* Let  $(S_i, \pi_{ji})$  be an inverse system where each set  $S_i$  is a topological space and each transition map is continuous. Then the inverse limit topology on  $\varprojlim S_i$  is the induced topology inherited from the product topology on  $\prod_{i \in I} S_i$  which has a basis consisting of the sets of the form  $\prod_{i \in I_i} U_i$ , with  $U_i$  is an open subset of  $S_i$  for every  $i \in I$  and  $U_i = S_i$  for all but a finitely many  $i \in I$ .

**Remark 2.17** The inverse limit topology is the smallest topology that makes each projection map  $\pi_i: \varprojlim S_i \rightarrow S_i$  continuous. A basis for the topology of  $S = \varprojlim S_i$  is given by the sets  $\pi_i^{-1}(U_i)$ , as  $i$  varies, where  $U_i$  is open in  $S_i$ .

**Remark 2.18** Let  $A$  be a subset of  $\varprojlim S_i$  and let  $A_i$  be the image of  $A$  under the canonical projection  $\pi_i : \varprojlim S_i \rightarrow S_i$ . Then the closure of  $A$  is given by  $\overline{A} = \varprojlim \overline{A_i}$ . In particular, if  $A$  is closed in  $\varprojlim S_i$ , then  $A = \overline{A} = \varprojlim \overline{A_i} = \varprojlim A_i = \cap(A + \text{Ker } \pi_i)$ . Moreover, if  $A = \varprojlim C_i$  where  $(C_i)$  is a closed inverse subsystem of  $(S_i)$ , then  $A$  is closed in  $\varprojlim S_i$ . If each  $S_i$  is given the discrete topology, the resulting inverse limit topology on  $\varprojlim S_i$  is called the prodiscrete topology. Let  $(S_i, \pi_{ji})$  and  $(S_i, \pi'_{ji})$  be two inverse systems over the same index set  $I$ . If for every  $i \in I$ , the map  $f_i : S_i \rightarrow S'_i$  is continuous, then  $\varprojlim f_i : \varprojlim S_i \rightarrow \varprojlim S'_i$  is also continuous. Moreover, if each  $S_i$  is a topological group and each transition map is a continuous homomorphism, then  $\varprojlim S_i$  is a topological group.

**Definition 2.19** (The Coset Topology on Algebraic Groups) Let  $G$  be an algebraic group (over an algebraically closed field). The closed sets in the coset topology are the subsets of  $G$  that are finite unions of cosets  $xH$ , where  $x$  ranges over  $G$  and  $H$  ranges over the set of closed algebraic subgroups of  $G$ .

**Definition 2.20** (The Coset Topology on Vector Spaces) Let  $V$  be a finite dimensional vector space over an arbitrary field  $F$ . There is a topology on  $V$  whose closed sets are the finite unions of sets of the form  $x + W$ , where  $x$  ranges over  $V$  and  $W$  ranges over the subspaces of  $V$ .

### 3 Hochschild-Mostow Inverse Systems

**Definition 3.1** (Essential Elements and Perfect Inverse Systems) Let  $(X, \pi)$  be an inverse system and let  $x_i \in X_i$ . Then

1.  $x_i$  is called an essential element if  $x_i \in \text{Image}(\pi_{ji})$  for all  $j \geq i$  in  $I$ . That is,  $x_i \in \cap_{j \geq i} \pi_{ji}(X_j)$
2.  $x_i$  is called super essential element if  $x_i$  can be completed to a full compatible family. That is, there exists  $x = (x_i) \in \varprojlim X_i$  where  $x_i$  is a component of  $x$
3.  $(X, \pi)$  is called a perfect inverse system if for all  $x_i \in X_i$ ,  $x_i$  is a component of a compatible family  $(x_i)_{i \in I} \in \varprojlim X_i$ , that is, each element  $x_i \in X_i$  is a super essential element, so each  $\pi_i : \varprojlim X_i \rightarrow X_i$  is surjective

**Remark 3.2** If  $x_i$  is a super essential element, then it is an essential element.

*Proof.* Let  $j \geq i$ , and let  $x = (x_i) \in \varprojlim X_i$  be a compatible family with  $x_i$  being one of its components. Then there exists  $x_j$  component of  $x$  such that  $\pi_{ji}(x_j) = x_i$ , so  $x_i \in \pi_{ji}(X_j)$ .

**Note 3.3** *The converse is not true in general.*

Take for instance the Waterhouse example. It provides an example where we have a surjective inverse system with empty inverse limit. Thus all the elements are essential, but, none is a part of a compatible family in  $\varprojlim X_i$ .

However, the converse is true when:

1.  $(X, \pi)$  is a surjective inverse sequence.
2. In the standard limit theorem situation (Theorem 3.6).
3. In the projective limit theorem situation (Theorem 3.14).

**Remark 3.4** *In a surjective inverse system, all the elements are essential.*

**Remark 3.5** *If  $(X, \pi)$  is a perfect inverse system then it is a surjective inverse system, while the converse is not true by the Waterhouse example.*

**Theorem 3.6** *(The Standard Limit Theorem) Let  $(S, \pi)$  be an inverse system (over any directed set  $I$ ) of non-empty compact Hausdorff spaces with continuous transition maps  $\pi_{ji}$ . Then*

1.  $S = \varprojlim S_i$  is a non-empty compact space
2. If each  $\pi_{ji}$  is surjective, then each  $\pi_i: \varprojlim S_i \rightarrow S_i$  is surjective
3. If  $s_i \in S_i$  is an essential element, then  $s_i$  is super essential

**Corollary 3.7** *Let  $(S, \pi)$  be an inverse system over any directed set  $I$  of non-empty finite sets. Then  $\varprojlim S_i$  is not empty, and if each  $\pi_{ji}$  is surjective, then each  $\pi_i: \varprojlim S_i \rightarrow S_i$  is surjective.*

**Definition 3.8** *(Hochschild-Mostow Inverse System) An inverse system  $(X, \pi)$  of non-empty sets is called a super inverse system, or a Hochschild-Mostow inverse system, or a compact system if each  $X_i$  can be equipped with a compact  $T_1$  topology such that each transition map is a closed continuous map.*

**Example 3.9** *Any inverse system of non-empty compact Hausdorff  $(T_2)$  spaces (with continuous transition maps) is a super inverse system. This is because every continuous map between compact Hausdorff spaces is a closed map.*

**Definition 3.10** *An inverse system  $(G, \pi)$  of groups is called  $G$ -compact if*

1.  $(G_i)$  is a compact inverse system
2. the left transition maps in each  $G_i$  are continuous.

**Example 3.11** Every inverse system of finite groups is  $G$ -compact.

**Example 3.12** Every inverse system of linear algebraic groups (over a fixed algebraically closed field)(whose transition maps are rational homomorphisms) is a  $G$ -compact inverse system provided we give each algebraic group its coset topology.

**Example 3.13** Every inverse system of finite dimensional vector spaces over any field with linear transition maps is a  $G$ -compact inverse system provided we give each vector space its coset topology.

**Theorem 3.14** (The Projective Limit Theorem) Let  $(V_i, \pi_{ji})$  be a super inverse system consisting of non-empty compact  $T_1$  spaces and continuous closed maps  $\pi_{ji} : V_j \rightarrow V_i$ . Then

1.  $\varprojlim V_i \neq \emptyset$ .
2. if each  $\pi_{ji}$  is surjective, then each canonical map projection  $\pi_i : \varprojlim V_i \rightarrow V_i$  is surjective.
3. if  $v_i \in V_i$  is an essential element, then  $v_i$  is super essential

**Remark 3.15** Let  $(V_i, \pi_{ji})$  be a compact inverse system and let  $(A_i)$  be a subinverse system of  $(V_i)$  such that each  $A_i$  is a non-empty closed subset of  $V_i$ . Then  $(A_i)$  is also a compact inverse system.

**Proposition 3.16** Let  $f: X \rightarrow Y$  be a continuous map of topological spaces. If  $X$  is compact,  $Y$  is a  $T_1$  space,  $\{A_i\}$  is a chain of closed subsets of  $X$ , then  $f(\cap_{i \in I} A_i) = \cap_{i \in I} f(A_i)$ .

*Proof.* It is obvious that  $f(\cap_{i \in I} A_i) \subseteq \cap_{i \in I} f(A_i)$ . On the other hand, let  $y \in \cap_{i \in I} f(A_i)$  so  $y = f(x_i)$  with  $x_i \in A_i$  for all  $i \in I$ . Then,  $y = f(x)$ , where  $x \in A_i$  for all  $i \in I$ . Consider the collection  $\{f^{-1}(y) \cap A_i\}_{i \in I}$ ; Since  $\{y\}$  is closed in the  $T_1$  space  $Y$  and  $f$  is continuous then  $f^{-1}(y)$  is closed in  $X$ . Hence  $f^{-1}(y) \cap A_i$  is closed in  $X$  for all  $i \in I$ . Also,  $x_i \in f^{-1}(y) \cap A_i$  for all  $i \in I$ . Since  $\{A_i\}$  is a chain,  $\{f^{-1}(y) \cap A_i\}$  is a chain of non-empty closed subsets of  $X$ . Take a finite subset  $\{f^{-1}(y) \cap A_{ij}\}_{j=1,2,\dots,n}$  of  $\{f^{-1}(y) \cap A_i\}$ . Because we have a chain, all elements are comparable, so there exists a smallest  $f^{-1}(y) \cap A_{il}$  for some  $l \in \{1, 2, \dots, n\}$ . So  $\bigcap_{j=1}^n (f^{-1}(y) \cap A_{ij}) = f^{-1}(y) \cap A_{il}$  is non-empty, so the finite intersection property holds for the collection  $\{f^{-1}(y) \cap A_i\}$ , but  $X$  is compact, then the intersection of the whole collection is non-empty, i.e. there exists  $x \in f^{-1}(y) \cap A_i$  for all  $i \in I$  such that  $y = f(x)$  and  $x \in \cap_{i \in I} A_i$ . Thus  $f(x) \in f(\cap_{i \in I} A_i)$  which implies that  $y \in f(\cap_{i \in I} A_i)$  and  $\cap_{i \in I} f(A_i) \subseteq f(\cap_{i \in I} A_i)$ .

## 4 Inverse Limits of Hochschild-Mostow Inverse Systems

**Theorem 4.1** *Let  $(B_i) \xrightarrow{g_i} (C_i)$  be a mapping of inverse systems of groups such that*

1. *each  $g_i$  is surjective, and*
2. *the subinverse system  $(Kerg_i)$  is a  $G$ -compact inverse system.*

*Then the limiting map  $\varprojlim g_i$  is surjective.*

*Proof.* Let  $c = (c_i)$  be an element of  $(C_i)$ . For each  $i \in I$ , let  $X_i = g_i^{-1}(c_i)$  and let  $A_i = Kerg_i$ . Each  $X_i$  is nonempty because each  $g_i$  is surjective. Moreover,  $(X_i)$  is a subinverse system of  $(B_i)$ . Now, each  $X_i$  is a left coset mod  $A_i$  and so we may transport the topology on  $A_i$  to  $X_i$  and the resulting topology on  $X_i$  is independent of the choice of  $b_i \in X_i$  since  $(A_i)$  is a  $G$ -compact inverse system. Now, for  $j \succ i$ , the maps  $X_j \rightarrow X_i$  are continuous and closed so  $\varprojlim X_i \neq \emptyset$  by the Projective limit theorem. Thus any element  $(b_i) \in \varprojlim X_i$  is sent to  $c = (c_i)$  by the limiting map  $\varprojlim g_i$ .

From the above theorem and because the inverse limit functor on groups is left exact, the following corollaries follow directly:

**Corollary 4.2** *Let  $1 \rightarrow (G_i) \xrightarrow{f_i} (G'_i) \xrightarrow{g_i} (G_i)'' \rightarrow 1$*

*be an exact sequence of inverse systems of groups such that the inverse system  $(G_i)$  is  $G$ -compact. Then the induced sequence of inverse limits  $1 \rightarrow \varprojlim(G_i) \xrightarrow{f} \varprojlim(G'_i) \xrightarrow{g} \varprojlim(G_i)'' \rightarrow 1$  is exact*

**Corollary 4.3** *Let  $0 \rightarrow (G_i) \xrightarrow{f_i} (G'_i) \xrightarrow{g_i} (G_i)'' \rightarrow 0$*

*be an exact sequence of finite dimensional vector spaces over an arbitrary field. Then the induced sequence of inverse limits*

$0 \rightarrow \varprojlim(G_i) \xrightarrow{f} \varprojlim(G'_i) \xrightarrow{g} \varprojlim(G_i)'' \rightarrow 0$   
*is exact*

**Corollary 4.4** *Let  $1 \rightarrow (G_i) \xrightarrow{f_i} (G'_i) \xrightarrow{g_i} (G_i)'' \rightarrow 1$*

*be an exact sequence of inverse systems of groups such that  $(G_i)$  is an inverse system of linear algebraic groups(over an algebraically closed field). Then the induced sequence of inverse limits*

$1 \rightarrow \varprojlim(G_i) \xrightarrow{f} \varprojlim(G'_i) \xrightarrow{g} \varprojlim(G_i)'' \rightarrow 1$   
*is exact*

**Theorem 4.5** *Let  $(V_i, \pi_{ji})$  be a Hochschild-Mostow inverse system and let  $V = \varprojlim V_i$  (with its inverse limit topology). Then each projection  $\pi_i : V \rightarrow V_i$  is a closed map.*

*Proof.* Let  $A$  be a closed subset of  $V$ . Then  $A = \varprojlim A_i = \varprojlim \overline{A_i}$  where  $A_i = \pi_i(A)$ . Because the  $\pi_{ji}$  are continuous,  $\pi_{ji}(\overline{A_j}) \subset \overline{\pi_{ji}(A_j)} = \overline{A_i}$  for all  $j \geq i$ , but the  $\pi_{ji}$  are also closed, so  $\pi_{ji}(\overline{A_j}) = \overline{A_i}$  and consequently  $(\overline{A_i})$  is a surjective super subinverse system of  $(V_i)$ . Now, by the Projective limit theorem, each projection  $\pi_i : \varprojlim \overline{A_i} \rightarrow \overline{A_i}$  is surjective.

But  $\varprojlim \overline{A_i} = A$  hence  $\pi_i(A) = \pi_i(\varprojlim \overline{A_i}) = \overline{A_i}$ . Thus,  $\pi_i$  is a closed map.

**Theorem 4.6** *Let  $(V_i, \pi_{ji})$  be a Hochschild-Mostow inverse systems system. Then,  $\varprojlim V_i$  (with its inverse limit topology) is compact.*

*Proof.* This theorem was proved by Stone in [7]. But here we give the following different proof: Let  $\{F_\alpha\}_{\alpha \in A}$  be a family of non-empty closed subsets of  $\varprojlim V_i$  with the finite intersection property. Then  $\bigcap_{\alpha \in A} F_\alpha \neq \emptyset$ :  $\pi_i(F_\alpha)$  is closed for all  $\alpha \in A$  since each  $\pi_i$  is a closed map. Let  $X_i = \bigcap_{\alpha \in A} \pi_i(F_\alpha)$ . Since each  $V_i$  is compact and each  $X_i$  is a closed subset of  $V_i$ , then each  $X_i$  is a non-empty compact  $T_1$  space. Hence  $(X_i)$  is a super inverse subsystem of  $(V_i)$ , and the Projective limit theorem implies that  $\varprojlim X_i \neq \emptyset$ . Let  $(x) \in \varprojlim X_i$ , and  $x_i = \pi_i((x)) \in X_i$ .  $x_i \in \pi_i(F_\alpha)$  for all  $i \in I$ ,  $\alpha \in A$ , hence  $(x) \in \varprojlim \pi_i(F_\alpha) = F_\alpha$  for all  $\alpha \in A$  (since each  $F_\alpha$  is closed). So  $(x) \in \bigcap_{\alpha \in A} (F_\alpha)$ , and thus the intersection of the whole collection is non-empty, and consequently  $\varprojlim V_i$  is compact.

**Remark 4.7** *One might expect that, we could prove the following:*

*"Let  $I$  be a chain, and let  $(X_i, \pi_{ji})_{i \in I}$  be a surjective inverse system of compact  $T_1$  spaces. Then  $\varprojlim X_i$  is compact". This, however, is false as the following example shows.*

**Example 4.8** *For  $n \in \mathbb{N}$ , define  $X_n = \mathbb{N}$ . Give each  $X_n$  the following topology: Each of the points  $1, 2, \dots, n$  is isolated, and for each  $m > n$ , a neighborhood base at  $m$  consists of the cofinite subsets of  $\mathbb{N}$  that contain  $m$ . This is easily seen to give a compact  $T_1$  topology. Define  $\pi_{n+1, n} : X_{n+1} \rightarrow X_n$  to be the identity map. This is continuous, because the only point at which the topologies of  $X_n$  and  $X_{n+1}$  differ is  $n + 1$ , which is isolated in  $X_{n+1}$ . The inverse limit  $\varprojlim X_i$  is the subset of  $\mathbb{N}^{\mathbb{N}}$  consisting of the points all of whose coordinates are equal. This is an infinite discrete set, since, given  $p = (m, m, \dots)$  of  $\varprojlim X_i$ , taken  $n > m$ , then  $\pi_n^{-1}(m)$  is a neighborhood of  $p$  in  $\mathbb{N}^{\mathbb{N}}$  that contains no other point of  $\varprojlim X_i$ , where  $\pi_n$  is the projection map. Thus  $\varprojlim X_i$  is not compact.*

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