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Fuzzy Detour Boundary Vertices in Fuzzy Graphs

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Abstract

A vertex v in a connected fuzzy graph G is a fuzzy detour boundary vertex of a vertex u, if $\Delta(u,w) \le \Delta(u,v)$ for every fuzzy detour neighbour w of v. This paper is devoted to see how the fuzzy detour boundary vertices are related to fuzzy detour interior vertices. Further, the relation between the fuzzy detour boundary vertices and fuzzy detour eccentric vertices are also studied. Some characterizations are presented for fuzzy detour boundary vertices.

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1. Introduction

The concept of fuzzy relations was introduced by Zadeh (1965) and these have found many applications in the analysis of cluster patterns, communication of information and abstractions. This theory proposed making the grade of membership of an element in a subset of a universal set, a value in the closed interval [0, 1] of real numbers. Rosenfeld (1975) considered fuzzy relations on fuzzy sets and developed the structure of fuzzy graphs obtaining analogues of several graph theoretical concepts such as paths, cycles and connectedness. Fuzzy graphs are finding an increasing number of applications in modeling real systems. Fuzzy models are becoming useful because of their aim in reducing the difference between the traditional numerical methods used in engineering and sciences and the symbolic models used in expert systems.

A new idea of distance called fuzzy detour μ-distance is introduced in [3]. Based on this concept a number of properties have been found [3, 4]. This concept has been tried to introduce the concepts of fuzzy detour neighbours and fuzzy detour boundary vertices.

Our goal here is to examine the characteristics of fuzzy detour boundary vertices. It is seen by an example that a fuzzy cut vertex may or may not be a fuzzy detour boundary vertex in fuzzy graphs in contrast to the concept that no cut vertex is a boundary vertex in non-fuzzy graphs. The fuzzy detour μ-distance from u to a fuzzy detour boundary vertex v of u attains the local maximum property. Equivalently, a vertex v is a fuzzy detour boundary vertex of u if no u-v fuzzy detour can be extended at v to a longer fuzzy detour. That is, beginning with a vertex u, a fuzzy detour boundary vertex v of u is reached when, locally, we cannot move farther from u.

2. Preliminaries

We summarize some basic definitions, most of which can be found in [5]. A fuzzy relation on a set S is a map μ: SxS $\rightarrow$ [0,1]. A fuzzy graph G:(σ,μ) is a pair of maps σ: S $\rightarrow$ [0,1] and μ:SxS $\rightarrow$ [0,1] such that $\mu(x,y)\leq \sigma(x)\wedge\sigma(y)$ where $\sigma(x)\wedge\sigma(y)$ denotes the minimum of $\sigma(x)$ and $\sigma(y)$. We call σ the fuzzy vertex set of G and μ the fuzzy arc set of G respectively. We assume that $\mu(x, y) = \mu(y,x)$ and $\mu(x,x)=0$ for all $x\in S$. We shall denote Max ($\sigma(x)$, $\sigma(y)$) by $\sigma(x)\vee\sigma(y)$.

A path from x to y in a fuzzy graph G:(σ,μ) is a sequence ρ: $x = x_0,x_1,x_2,\ldots..x_n=y$ of distinct nodes such that $\sigma(x_i)>0$ and $\sigma(x_{i-1})\wedge\sigma(x_i) \geq\mu(x_{i-1},x_i)>0$. Then $\Lambda_i\ \mu(x_{i-1},x_i)$ is called the strength of ρ. The maximum of the strengths of all paths in G from x to y is called $\mu^{\infty}(x,y)$ and we call G connected if $\mu^{\infty}(x,y) >0$ for all x,y in S. If H is connected and G is a full fuzzy sub graph of H then G is connected. A vertex is a fuzzy cut vertex of G:(σ,μ) if removal of it reduces the strength of connectedness between some other pair of vertices. Equivalently, w is a fuzzy cut vertex if and only if there exist u, v distinct from w such that w is on every strongest u-v path. It is shown in [6] that every fuzzy graph has at least two vertices that are not fuzzy cut vertices.

In [3,4] the authors introduced the concept of fuzzy detour μ-distance and properties based on them.

Let G:(σ,μ) be a connected fuzzy graph. The fuzzy detour μ-distance $\Delta(u,v)$ between vertices u and v is defined to be the maximum μ-length of any u-v path, where the μ-length of a pathρ: $u_0 ,u_1,\ldots.,u_n$ is $l(\rho)=\sum_{i=1}^{n}\frac{1}{\mu(u_{i-1},u_i)}$

A u-v path of length Δ (u,v) is called u-v fuzzy detour. It is seen that the fuzzy detour μ-distance is a metric. The fuzzy detour μ- eccentricity of a vertex v of a fuzzy graph is the maximum fuzzy detour μ- distance from v to any vertex of G. It is obvious that $e_{\Delta}(v) \geq \Delta(u,v)$ for any vertex u of G. The fuzzy detour μ-radius of G is the minimum fuzzy detour μ- eccentricity among the vertices of G. The fuzzy detour μ - diameter of G is the maximum fuzzy detour μ- eccentricity among the vertices of G. As with μ-distance, fuzzy detour μ-distance is also a metric on the vertex set of every connected fuzzy graph.

3. Definitions and Results

3.1 Definition: Let G be a nontrivial connected fuzzy graph G: (σ, μ). For a vertex v in G, define $\Delta^{-}(v) = \min\{\Delta(u,v) : u \in V(G) - \{v\}\}$
A vertex u (≠ v) is called a fuzzy detour neighbor of v if $\Delta(u,v) = \Delta^{-}(v)$. The fuzzy detour neighbours of v are denoted by $N_{\Delta}(v)$.
If v is a fuzzy detour eccentric vertex of u in a connected fuzzy graph G, then no vertex of G is farther, in the fuzzy detour sense, from u than v is. In particular, if w is a fuzzy detour neighbor of v, then Δ (u, w) ≤ Δ (u, v). This property can be used to define the following.

3.2 Definition: A vertex v in a connected fuzzy graph G is a fuzzy detour boundary vertex of a vertex u if Δ (u, w) ≤ Δ (u, v) for every fuzzy detour neighbour w of v.

A vertex v is a fuzzy detour boundary vertex of a fuzzy graph G if v is a fuzzy detour boundary of some vertex of G.

3.3 Example

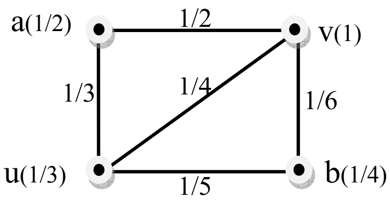

Fig. 1 Fuzzy detour boundary vertices in a fuzzy graph

For the fuzzy graph G of Fig. 1, the vertices u and b are fuzzy detour neighbours of a, v is a fuzzy detour neighbour of b; b and v are fuzzy detour neighbours of u and b is the fuzzy detour neighbour of v; a is the fuzzy detour boundary vertex of v, b is a fuzzy detour boundary vertex of u, v is the fuzzy detour boundary vertex of a and u but u is not the fuzzy detour boundary vertex of a. Therefore a, b and v are fuzzy detour boundary vertices of the fuzzy graph G. Also a and u are fuzzy cut vertices of G and they are also fuzzy detour boundary vertices of G. In a crisp graph no cut vertex of a connected graph G is a boundary vertex of G but in fuzzy graphs a fuzzy cut vertex may or may not be a fuzzy detour boundary vertex.

Explanation:

Fuzzy Detour Distances

Δ (a, b) =13, a-u-v-b Δ (a, u) = 13, a-v-b-u Δ (a, v) = 14, a-u-b-v
Δ (b, u) = 11, b-v-a-u Δ (b, v) = 10, b-u-a-v Δ (u, v) = 11, u-b-v

Fuzzy Detour Neighbours

Δ^{-} (a) = 13 = Δ (a, b) = Δ (a, u); u, b are fuzzy detour neighbours of a.
Δ^{-} (b) = 10 = Δ (b, v); v is a fuzzy detour neighbour of b.
Δ^{-} (u) = 11 = Δ (u, b) = Δ (u, v); b and v are fuzzy detour neighbours of u.
Δ^{-} (v) = 10 = Δ (v, b); b is a fuzzy detour neighbour of v.

Fuzzy Detour Boundary Vertices

Δ(v,u) ≤ Δ(v,a) and Δ(v,b)≤Δ(v,a). Therefore a is a fuzzy detour boundary vertex of v. Δ (u, v) = Δ (u, b). Therefore b is a fuzzy detour boundary vertex of u.
Δ (a, b) ≤ Δ (a, v) and Δ (u, b) ≤ Δ (u, v). Therefore v is a fuzzy detour boundary vertex of u and a.Δ (a, b) ≤ Δ (a, u) and Δ (a, v) not less than or equal to Δ (a, u). Therefore u is not a fuzzy detour boundary vertex of a, for its every fuzzy detour neighbor b and v.

3.4 Example

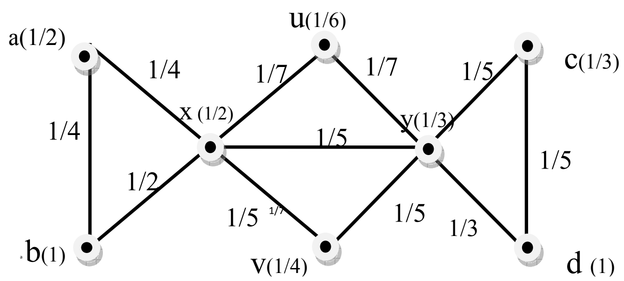

Fig. 2 Fuzzy Detour Boundary Vertices in a Fuzzy Graph

Explanation:

Fuzzy Detour Distances

Δ(a,b)=6, Δ(a,c)=28, Δ(a,d)=30, Δ(a,x)=6, Δ(a,y)=20, Δ(a,u)=23, Δ(a,v)=25; Δ(b,c)=30, Δ(b,d)=32, Δ(b,x)=8, Δ(b,y)=22, Δ(b,u)=25, Δ(b,v)=27; Δ(c,v)=27, Δ(c,y)=8, Δ(c,u)=25, Δ(c,x)=22, Δ(c,d)=8, Δ(d,x)=24, Δ(d,y)=27, Δ(d,u)=27, Δ(d,v)=29, Δ(x,y)=14, Δ(x,u)=17, Δ(x,v)=19; Δ(y,u)=17, Δ(y,v)=19; Δ(u,v)=17.

Fuzzy Detour Neighbours

Δ^{-}(a)=6=Δ(a,x)=Δ(a,b) so that b,x are fuzzy detour neighbours of a; Δ^{-}(b)=8=Δ(a,b) so that a is a fuzzy detour neighbour of b; Δ^{-}(c)=8=Δ(c,d) =Δ(c,y) so that d,y are fuzzy detour neighbours of c; Δ^{-}(d)=8=Δ(c,d) so that c is a fuzzy detour neighbour of d; Δ^{-}(x)=6=Δ(a,x) so that a is a fuzzy detour neighbour of x; Δ^{-}(y) = 8 = Δ(c,y) so that c is a fuzzy detour neighbour of y ; Δ^{-}(u)=17=Δ(u,v)= Δ(y,u) so that y,v are fuzzy detour neighbours of u; Δ^{-}(v)=17=Δ(u,v) so that u is a fuzzy detour neighbour of v.

Fuzzy Detour Boundary Vertices

a,c and u are not fuzzy detour boundary vertices of the fuzzy graph G. b,d and v are fuzzy detour boundary vertices of every vertices of G. x and y are respectively the fuzzy detour boundary vertices of b and d only.

3.5 Theorem: Let G be a nontrivial connected fuzzy graph and let u be a vertex of G. Every vertex distinct from u is a fuzzy detour boundary vertex of u if and only if e_Δ (u) = a, $1 \leq a < \infty$.

Proof:

Assume first that e_Δ (u) = a and let v be a vertex of G distinct from u such that Δ (u, v) = a. That is, v is the farthest vertex from u in the fuzzy detour sense. Let w be a fuzzy detour neighbour of v, so that Δ (u, w) $\leq$ a. Therefore Δ (u, w) $\leq \Delta$ (u, v) and hence v is a fuzzy detour boundary vertex of u.

Conversely, we assume, to the contrary, that every vertex of G different from u is a fuzzy detour boundary vertex of u but e_Δ (u) > a. Let x be a fuzzy detour boundary vertex such that e_Δ (u) = Δ (u, x) = a + k where k is such that Δ (u, x) = k for some y in G. Also we assume that u is a fuzzy detour neighbour of y in G. Therefore, Δ (u, y) = k < a + k = Δ (u, x). That is, Δ (u, x) > Δ (u, y). This shows that y is not a fuzzy detour boundary vertex of G.

There are certain vertices in a connected fuzzy graph G that have a close connection with fuzzy detour boundary vertices. A vertex y distinct from x and z is said to lie between x and z if Δ (u, z) = Δ(x, y) + Δ(y, z). That is, the triangle inequality becomes equality. This concept has been seen in the following definition.

3.6 Definition: A vertex v is a fuzzy detour interior vertex of G if for every vertex u distinct from v, there exists a vertex w such that v lies on the fuzzy detour from u to w. The fuzzy detour interior Int_Δ (G) of G is the fuzzy sub graph of G induced by fuzzy detour interior vertices.

We now see that the fuzzy detour interior vertices are precisely those vertices that are not fuzzy detour boundary vertices.

3.7 Theorem: Let G be a connected fuzzy graph. A vertex v is a fuzzy detour boundary vertex of G if and only if v is not a fuzzy detour interior vertex of G.

Proof:

Let v be a fuzzy detour boundary vertex of G and v is a fuzzy detour boundary vertex of the vertex u. Assume, to the contrary, that v is a fuzzy detour interior vertex of G. Therefore there exists a vertex w distinct from u and v such that v lies on the fuzzy detour from u to w.

Let P : u = v_1, v_2, v_3,..., v_i = v , v_{i+1} ,…, v_j = w , 1 < i < j , be the u to w fuzzy detour. Let v_{i+1} be the fuzzy detour neighbour of v. That is, $\Delta(u, v_{i+1}) \leq \Delta(u,v)$. Now $\Delta(u,v_{i+1}) = \Delta(u,v) + \Delta(v,v_{i+1}) = \Delta(u,v) + k$, $1 \leq k < \infty$; that is, $\Delta(u, v_{i+1}) > \Delta(u,v)$, a contradiction. Therefore v is not a fuzzy detour interior vertex.

Conversely, assume that v is a vertex that is not a fuzzy detour interior vertex of G. Then there exists some vertex u such that for every vertex w distinct from u and v, the vertex v does not lie on the u-v fuzzy detour. Let x be a fuzzy detour neighbour of v.

Then, $\Delta(u,x) \leq \Delta(u,v) + \Delta(v,x) = \Delta(u,v) + k$, where $\Delta(v,x) = k$ and $1 \leq k < \infty$.

Since v does not lie on the u-x fuzzy detour we must have $\Delta(u, x) < \Delta(u, v) + k$. That is, $\Delta(u, x) \leq \Delta(u, v)$. That is v is a fuzzy detour boundary vertex of u.

3.8 Theorem: Let v be a vertex in a connected fuzzy graph G such that v belongs to a block B and v is not fuzzy cut vertex of G. Then v is a fuzzy detour boundary vertex of G if and only if v is a fuzzy detour boundary vertex of B.

Proof:

Let v be a fuzzy detour boundary vertex of a block B. Therefore v is a fuzzy detour boundary vertex of some vertex of B. Since B is in G, v is a fuzzy detour boundary vertex of G.

Conversely, let v be a fuzzy detour boundary vertex of G. Therefore v is a fuzzy detour boundary vertex of some vertex w of G. Since v is not a fuzzy cut vertex it belongs to a unique block B of G. If $w \in B$, then the proof is complete. Thus, we may assume that $w \in B' \neq B$. For each y in B, the w-y fuzzy detour must pass through the fuzzy cut vertex x common to both B and B'. Therefore, $\Delta(w, v) = \Delta(w, x) + \Delta(x, v)$.Let u be a fuzzy detour neighbour of v. Then $u \in B$ and so $\Delta(w,u) = \Delta(w,x) + \Delta(x,u)$, Since v is a fuzzy detour boundary vertex of w, $\Delta(w,u) \leq \Delta(w,v)$, we have from the above relations $\Delta(x,u) \leq \Delta(x,v)$. That is, v is a boundary vertex of x and hence of B.

3.9 Theorem: Let G be a connected fuzzy graph with fuzzy detour diameter b. Then every vertex v is a fuzzy detour boundary vertex of G unless v is the unique vertex of G having fuzzy detour eccentricity less than b.

Proof:

Let v be a vertex in G. If $e_\Delta(v)$ not less than b, then $e_\Delta(v) = b$, since $diam_\Delta(G) = b$. Thus there is a vertex u such that $\Delta(u, v) = b$. Since $\Delta(u, w) \leq b$ for all $w \in N_\Delta(v)$, it follows that v is a fuzzy detour boundary vertex of u and so v is a fuzzy detour boundary vertex of G.

Suppose that v is the unique vertex of G having fuzzy detour eccentricity less than b. We claim that v is not the fuzzy detour boundary vertex of G. Assume, to the contrary, that v is a fuzzy detour boundary vertex of some vertex w, so that $\Delta(v,w) < b$. Since v is unique, $e_\Delta(w) \nless b$. Then there exists a vertex u that is a fuzzy detour neighbour of v so that $\Delta(u,w) \nless b$, otherwise, u and v are fuzzy detour boundary vertices, contradicting that v is unique. That is, we have $\Delta(w,v) < \Delta(u,w)$, contradicting the assumption that v is a fuzzy detour boundary vertex of some vertex w.

4. Conclusion

The properties we have seen above may help to apply to real life problems. Certain emergency facilities may be available at some locations in a city. These will help for utilization to its neighbouring places. The facilities may be identified as fuzzy detour boundary vertices whereas the neighbouring places are considered to be fuzzy detour neighbours. If a city is modeled as self detour boundary graph then the entire city is equipped with uniform distribution of emergency facilities.

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