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On Weyl Fractional Derivatives of the Product of Hypergeometric Function and H-Function

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Abstract

The aim of this paper is to establish a theorem on Weyl fractional derivatives of the product of hypergeometric function and the H-function. Certain special cases of our theorem have also been discussed.

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1 Introduction

Our purpose of this paper is to develop a theorem on Weyl fractional derivatives of the product of hypergeometric function and the H-function. The results derived in this paper provide an extension of our work [7].

Let A denote a class of good functions. By a good function f, we mean [5] a function which is everywhere differentiable any number of times and if it and all of its derivatives are $O(x^{-\nu})$, for all ν as x increases without limit.

We define the Weyl fractional derivatives of a function $g(z)$ as follows: Let $g \in A$, then for $q < 0$,

$$ {}_zW_\infty^q g(z) = \frac{(-1)^q}{\Gamma(-q)} \int_z^\infty (u-z)^{-q-1} g(u)du. \tag{1}$$

For $q \geq 0$,

$$ {}_zW_\infty^q g(z) = \frac{d^n}{dz^n}({}_zW_\infty^{q-n} g(z)), \tag{2}$$

n being a positive integer such that $n > q$.

We recall the definition of Gauss's hypergeometric function in terms of Pochhammer symbol:

$$ {}_2F_1(a, b; c; x) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r} \frac{x^r}{r!} \tag{3} $$

The H-function, defined by Fox [2], in terms of Mellin-Barnes type contour integral as follows:

$$ H_{P,Q}^{M,N}\left[z \middle| \begin{matrix} (a_j, e_j)_{1,P} \\ (b_j, f_j)_{1,Q} \end{matrix} \right] = \frac{1}{2\pi i} \int_L \theta(s) z^s ds, \tag{4} $$

where, for convenience,

$$ \theta(s) = \frac{\prod_{j=1}^{M} \Gamma(b_j - f_j s) \prod_{j=1}^{N} \Gamma(1 - a_j + e_j s)}{\prod_{j=M+1}^{Q} \Gamma(1 - b_j + f_j s) \prod_{j=N+1}^{P} \Gamma(a_j - e_j s)}, \tag{5} $$

$z \neq 0$, and an empty product is interpreted as unity. The integers M, N, P, Q are such that $0 \le N \le P$, $0 \le M \le Q$; the coefficients $e_j (j = 1, \ldots, P)$, $f_j (j = 1, \ldots, Q)$ are all positive; $a_j (j = 1, \ldots, P)$, $b_j (j = 1, \ldots, Q)$ are complex numbers. L is a suitably chosen contour such that all the poles of $\theta(s)$ are simple. Owing to the popularity of the special functions, those are defined in (3) and (4) (c.f. [6], [4]), details regarding these are avoided.

Braksma [1] has shown that the integral in the right side of (4) is absolutely convergent when $A > 0$ so that $|\arg z| < \frac{1}{2} A\pi$, where

$$ A = \sum_{j=1}^{N} e_j - \sum_{j=N+1}^{P} e_j + \sum_{j=1}^{M} f_j - \sum_{j=M+1}^{Q} f_j \tag{6} $$

2 Mathematical Prerequisites

To establish the main result, we need the following integral of the H-function [8]:

$$ \int_x^\infty t^{\rho-1} (t - x)^{\sigma-1} H_{P,Q}^{M,N}\left[z t^\mu (t - x)^\nu \middle| \begin{matrix} (a_j, e_j)_{1,P} \\ (b_j, f_j)_{1,Q} \end{matrix} \right] dt $$

$$ = x^{\rho+\sigma-1} H_{P+2,Q+1}^{M+1,N+1}\left[z x^{\mu+\nu} \middle| \begin{matrix} (1 - \sigma, \nu), (a_j, e_j)_{1,P}, (1 - \rho, \mu) \\ (1 - \rho - \sigma, \mu + \nu), (b_j, f_j)_{1,Q} \end{matrix} \right], \tag{7} $$

where,

(i) ρ, σ are complex numbers and μ, ν are positive real numbers,

(ii) $|\arg z| < \frac{1}{2}A\pi$, (A is defined in (6)),

(iii) $\min\left[Re\left(\frac{1-\rho-\sigma}{\mu+\nu}\right), \min_{1\leq j\leq M}\left[Re\left(\frac{b_j}{f_j}\right)\right]\right]$
$> \max\left[-Re\left(\frac{\sigma}{\nu}\right), \max_{1\leq j\leq N}\left[Re\left(\frac{a_j-1}{e_j}\right)\right]\right].$

The rth derivative formula for the H-function is given by [3]:

$$\frac{d^r}{dx^r}\left\{x^\lambda H_{P,Q}^{M,N}\left[ux^h\middle| \begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right]\right\}$$
$$= x^{\lambda-r}H_{P+1,Q+1}^{M,N+1}\left[ux^h\middle| \begin{matrix}(-\lambda,h),(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q},(-\lambda+r,h)\end{matrix}\right] \quad (8)$$

Lemma 2.1. *From Rainville [6], we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k) \quad (9)$$

3 Weyl fractional derivatives of the product of hypergeometric function and H-function

In this section, we propose to establish the following theorem concerning the Weyl fractional derivatives of the product of hypergeometric function and H-function:

Theorem 3.1. *Let M, N, P and Q be non negative integers such that $0 \leq M \leq Q$, $0 \leq N \leq P$ and $\sum_{j=1}^{N} e_j - \sum_{j=N+1}^{P} e_j + \sum_{j=1}^{M} f_j - \sum_{j=M+1}^{Q} f_j > 0$, together with the sets of conditions (i)-(iii) given with (7).*
Then, for all values of q,

$$_zW_\infty^q\left\{x^{\rho-1}(z-x)^{\sigma-1}e^{-xt}{}_2F_1(\alpha,\beta;\gamma;ax^\zeta)H_{P,Q}^{M,N}\left[yx^\mu(x-z)^\nu\middle| \begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right]\right\}$$
$$= \frac{(-1)^{q+\sigma-1}}{\Gamma(-q)}z^{\rho+\sigma-q-2}\,e^{-zt}\sum_{n=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{t^{n-k}}{(n-k)!}(-1)^{n-k}z^{(\zeta-1)k+n}$$
$$\times H_{P+2,Q+1}^{M+1,N+1}\left[yz^{\mu+\nu}\middle| \begin{matrix}(2+q-\sigma+k-n,\nu),(a_j,e_j)_{1,P},(1-\rho-\zeta k,\mu)\\(2+q-\rho-\sigma-(\zeta-1)k-n,\mu+\nu),(b_j,f_j)_{1,Q}\end{matrix}\right], \quad (10)$$

where,

$$f(k) = \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k}{k!}. \quad (11)$$

Proof. Denoting the left hand side expression of (10) by Ω, we have,

$$\begin{aligned}\Omega =& {}_zW^q_\infty\Big\{x^{\rho-1}(z-x)^{\sigma-1}e^{-xt}{}_2F_1(\alpha,\beta;\gamma;ax^\zeta)\\&\times H^{M,N}_{P,Q}\left[yx^\mu(x-z)^\nu\Big|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right]\Big\}\\=&{}_zW^q_\infty\Big\{e^{-zt}x^{\rho-1}(z-x)^{\sigma-1}e^{(z-x)t}{}_2F_1(\alpha,\beta;\gamma;ax^\zeta)\\&\times H^{M,N}_{P,Q}\left[yx^\mu(x-z)^\nu\Big|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right]\Big\}\end{aligned}$$

Now we replace $e^{(z-x)t}$ by $\sum_{n=0}^{\infty}\frac{(z-x)^nt^n}{n!}$ and express the hypergeometric function with the help of (3), to get

$$\begin{aligned}\Omega =& {}_zW^q_\infty\Big\{e^{-zt}x^{\rho-1}(z-x)^{\sigma-1}\sum_{n=0}^{\infty}\frac{(z-x)^nt^n}{n!}\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^kx^{\zeta k}}{k!}\\&\times H^{M,N}_{P,Q}\left[yx^\mu(x-z)^\nu\Big|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right]\Big\}\\=&{}_zW^q_\infty\Big\{e^{-zt}x^{\rho-1}(z-x)^{\sigma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^kx^{\zeta k}}{k!}\frac{(z-x)^nt^n}{n!}\\&\times H^{M,N}_{P,Q}\left[yx^\mu(x-z)^\nu\Big|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right]\Big\}\end{aligned}$$

Now using the result (9), the above expression reduces to

$$\begin{aligned}\Omega =& {}_zW^q_\infty\Big\{e^{-zt}x^{\rho-1}(z-x)^{\sigma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^kx^{\zeta k}}{k!}\frac{(z-x)^{n-k}t^{n-k}}{(n-k)!}\\&\times H^{M,N}_{P,Q}\left[yx^\mu(x-z)^\nu\Big|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right]\Big\}\\=&e^{-zt}\sum_{n=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{t^{n-k}}{(n-k)!}\,{}_zW^q_\infty\Big\{x^{\rho+\zeta k-1}(z-x)^{\sigma+n-k-1}\\&\times H^{M,N}_{P,Q}\left[yx^\mu(x-z)^\nu\Big|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right]\Big\},\end{aligned}\tag{12}$$

where $f(k)$ is given in (11).

The relation (12), due to (1), further reduces to

$$\Omega = e^{-zt}\sum_{n=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{t^{n-k}}{(n-k)!}\left\{\frac{(-1)^q}{\Gamma(-q)}\int_z^{\infty}(u-z)^{-q-1}u^{\rho+\zeta k-1}(z-u)^{\sigma+n-k-1}\right.$$
$$\left.\times H_{P,Q}^{M,N}\left[yu^{\mu}(u-z)^{\nu}\middle|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right]du\right\}$$
$$= e^{-zt}\sum_{n=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{t^{n-k}}{(n-k)!}\frac{(-1)^{q+\sigma+n-k-1}}{\Gamma(-q)}\int_z^{\infty}u^{\rho+\zeta k-1}(u-z)^{\sigma+n-k-q-2}$$
$$\times H_{P,Q}^{M,N}\left[yu^{\mu}(u-z)^{\nu}\middle|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right]du.$$

Now, employ (7) to evaluate the integral on the right hand side, we get

$$\Omega = e^{-zt}\sum_{n=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{t^{n-k}}{(n-k)!}\frac{(-1)^{q+\sigma+n-k-1}}{\Gamma(-q)}z^{\rho+\sigma+(\zeta-1)k+n-q-2}$$
$$\times H_{P+2,Q+1}^{M+1,N+1}\left[yz^{\mu+\nu}\middle|\begin{matrix}(2+q-\sigma+k-n,\nu),(a_j,e_j)_{1,P},(1-\rho-\zeta k,\mu)\\(2+q-\rho-\sigma-(\zeta-1)k-n,\mu+\nu),(b_j,f_j)_{1,Q}\end{matrix}\right]$$
$$=\frac{(-1)^{q+\sigma-1}}{\Gamma(-q)}z^{\rho+\sigma-q-2}\,e^{-zt}\sum_{n=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{t^{n-k}}{(n-k)!}(-1)^{n-k}z^{(\zeta-1)k+n}$$
$$\times H_{P+2,Q+1}^{M+1,N+1}\left[yz^{\mu+\nu}\middle|\begin{matrix}(2+q-\sigma+k-n,\nu),(a_j,e_j)_{1,P},(1-\rho-\zeta k,\mu)\\(2+q-\rho-\sigma-(\zeta-1)k-n,\mu+\nu),(b_j,f_j)_{1,Q}\end{matrix}\right].$$

For $q \geq 0$, invoking the definition (2), the relation (12), further reduces to

$$\Omega = e^{-zt}\sum_{n=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{t^{n-k}}{(n-k)!}\frac{d^r}{dz^r}\left\{{}_zW_{\infty}^{q-r}\left(x^{\rho+\zeta k-1}(z-x)^{\sigma+n-k-1}\right.\right.$$
$$\left.\left.\times H_{P,Q}^{M,N}\left[yx^{\mu}(x-z)^{\nu}\middle|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right]\right)\right\}$$
$$= e^{-zt}\sum_{n=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{t^{n-k}}{(n-k)!}\frac{d^r}{dz^r}\left\{\frac{(-)^{q-r}}{\Gamma(r-q)}\int_z^{\infty}(u-z)^{r-q-1}u^{\rho+\zeta k-1}\right.$$
$$\left.\times (z-u)^{\sigma+n-k-1}H_{P,Q}^{M,N}\left[yu^{\mu}(u-z)^{\nu}\middle|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right]du\right\}$$
$$= e^{-zt}\sum_{n=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{t^{n-k}}{(n-k)!}\frac{(-)^{q-r+\sigma+n-k-1}}{\Gamma(r-q)}\frac{d^r}{dz^r}\left\{\int_z^{\infty}u^{\rho+\zeta k-1}\right.$$
$$\left.\times (u-z)^{\sigma+r-q+n-k-2}H_{P,Q}^{M,N}\left[yu^{\mu}(u-z)^{\nu}\middle|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right]du\right\}$$

Now, evaluating the integral with the help of (7), we get

$$\Omega = e^{-zt}\sum_{n=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{t^{n-k}}{(n-k)!}\frac{(-)^{q-r+\sigma+n-k-1}}{\Gamma(r-q)}\frac{d^r}{dz^r}\left\{z^{\rho+\sigma+(\zeta-1)k+r-q+n-2}\right.$$

$$\times H^{M+1,N+1}_{P+2,Q+1}\left[yz^{\mu+\nu}\left|\begin{matrix}(2+q-\sigma-r+k-n,\nu),(a_j,e_j)_{1,P},(1-\rho-\zeta k,\mu)\\(2+q-\rho-\sigma-(\zeta-1)k-r-n,\mu+\nu),(b_j,f_j)_{1,Q}\end{matrix}\right.\right]\Bigg\}$$

Finally, we apply the r th derivative formula for the H-function [3], which is given in (8), to obtain

$$\Omega=e^{-zt}\sum_{n=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{t^{n-k}}{(n-k)!}\frac{(-1)^{q-r+\sigma+n-k-1}}{\Gamma(r-q)}z^{\rho+\sigma+(\zeta-1)k-q+n-2}$$

$$\times H^{M+1,N+2}_{P+3,Q+2}\left[yz^{\mu+\nu}\left|\begin{matrix}(2+q-\rho-\sigma-(\zeta-1)k-r-n,\mu+\nu),\\(2+q-\rho-\sigma-(\zeta-1)k-r-n,\mu+\nu),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(2+q-\sigma-r+k-n,\nu),(a_j,e_j)_{1,P},(1-\rho-\zeta k,\mu)\\(b_j,f_j)_{1,Q},(2+q-\rho-\sigma-(\zeta-1)k-n,\mu+\nu)\end{matrix}\right]$$

On replacement of $(q-r)$ by q, we may obtain

$$\begin{aligned}\Omega=&e^{-zt}\sum_{n=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{t^{n-k}}{(n-k)!}\frac{(-1)^{q+\sigma+n-k-1}}{\Gamma(-q)}z^{\rho+\sigma+(\zeta-1)k+n-q-2}\\&\times H^{M+1,N+1}_{P+2,Q+1}\left[yz^{\mu+\nu}\left|\begin{matrix}(2+q-\sigma+k-n,\nu),(a_j,e_j)_{1,P},(1-\rho-\zeta k,\mu)\\(2+q-\rho-\sigma-(\zeta-1)k-n,\mu+\nu),(b_j,f_j)_{1,Q}\end{matrix}\right.\right]\\=&\frac{(-1)^{q+\sigma-1}}{\Gamma(-q)}z^{\rho+\sigma-q-2}\,e^{-zt}\sum_{n=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{t^{n-k}}{(n-k)!}(-1)^{n-k}z^{(\zeta-1)k+n}\\&\times H^{M+1,N+1}_{P+2,Q+1}\left[yz^{\mu+\nu}\left|\begin{matrix}(2+q-\sigma+k-n,\nu),(a_j,e_j)_{1,P},(1-\rho-\zeta k,\mu)\\(2+q-\rho-\sigma-(\zeta-1)k-n,\mu+\nu),(b_j,f_j)_{1,Q}\end{matrix}\right.\right].\end{aligned}$$

□

4 Special Cases

On specializing the parameters of the hypergeometric function and H-function, the result in (10) leads to the following interesting results:

(i) Putting $t=0$ and $a=0$ in (10), the exponential function and the hypergeometric function reduces to unity and at the same time replacing ρ by $\lambda+1$ and σ by 1, (10) reduces to the result:

$$\begin{aligned}&{}_zW^q_\infty\left\{x^\lambda H^{M,N}_{P,Q}\left[yx^\mu(x-z)^\nu\left|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right.\right]\right\}\\=&\frac{(-1)^q}{\Gamma(-q)}z^{\lambda-q}H^{M+1,N+1}_{P+2,Q+1}\left[yz^{\mu+\nu}\left|\begin{matrix}(1+q,\nu),(a_j,e_j)_{1,P},(-\lambda,\mu)\\(q-\lambda,\mu+\nu),(b_j,f_j)_{1,Q}\end{matrix}\right.\right].\end{aligned}\quad(13)$$

(ii) Now replacing μ by $-\mu$ and ν by $-\nu$, the result in (13), further reduces to

$$\begin{aligned}&{}_zW_\infty^q\left\{x^\lambda H_{P,Q}^{M,N}\left[yx^{-\mu}(x-z)^{-\nu}\left|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right.\right]\right\}\\&=\frac{(-1)^q}{\Gamma(-q)}z^{\lambda-q}H_{P+1,Q+2}^{M+1,N+1}\left[yz^{-\mu-\nu}\left|\begin{matrix}(1-q+\lambda,\mu+\nu),(a_j,e_j)_{1,P}\\(-q,\nu),(b_j,f_j)_{1,Q},(1+\lambda,\mu)\end{matrix}\right.\right].\end{aligned}\tag{14}$$

(iii) Next replacing ν by $-\nu$, (13) correspond to the following results according as $\mu>\nu,\mu<\nu$ and $\mu=\nu$ respectively, that is,
for $\mu>\nu$,

$$\begin{aligned}&{}_zW_\infty^q\left\{x^\lambda H_{P,Q}^{M,N}\left[yx^{\mu}(x-z)^{-\nu}\left|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right.\right]\right\}\\&=\frac{(-1)^q}{\Gamma(-q)}z^{\lambda-q}H_{P+1,Q+2}^{M+2,N}\left[yx^{\mu-\nu}\left|\begin{matrix}(a_j,e_j)_{1,P},(-\lambda,\mu)\\(-q,\nu),(q-\lambda,\mu-\nu),(b_j,f_j)_{1,Q}\end{matrix}\right.\right],\end{aligned}\tag{15}$$

for $\mu<\nu$,

$$\begin{aligned}&{}_zW_\infty^q\left\{x^\lambda H_{P,Q}^{M,N}\left[yx^{\mu}(x-z)^{-\nu}\left|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right.\right]\right\}\\&=\frac{(-1)^q}{\Gamma(-q)}z^{\lambda-q}H_{P+2,Q+1}^{M+1,N+1}\left[yx^{\mu-\nu}\left|\begin{matrix}(1-q+\lambda,\nu-\mu),(a_j,e_j)_{1,P},(-\lambda,\mu)\\(-q,\nu),(b_j,f_j)_{1,Q}\end{matrix}\right.\right],\end{aligned}\tag{16}$$

and for $\mu=\nu$,

$$\begin{aligned}&{}_zW_\infty^q\left\{x^\lambda H_{P,Q}^{M,N}\left[yx^{\mu}(x-z)^{-\nu}\left|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right.\right]\right\}\\&=\frac{(-1)^q\Gamma(q-\lambda)}{\Gamma(-q)}z^{\lambda-q}H_{P+1,Q+1}^{M+1,N}\left[y\left|\begin{matrix}(a_j,e_j)_{1,P},(-\lambda,\mu)\\(-q,\nu),(b_j,f_j)_{1,Q}\end{matrix}\right.\right].\end{aligned}\tag{17}$$

(iv) Finally, writing $-\mu$ instead of μ, (13) yields the following results according as $\mu>\nu,\mu<\nu$ and $\mu=\nu$ respectively, that is,
for $\mu>\nu$,

$$\begin{aligned}&{}_zW_\infty^q\left\{x^\lambda H_{P,Q}^{M,N}\left[yx^{-\mu}(x-z)^{\nu}\left|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right.\right]\right\}\\&=\frac{(-1)^q}{\Gamma(-q)}z^{\lambda-q}H_{P+2,Q+1}^{M,N+2}\left[yz^{-\mu+\nu}\left|\begin{matrix}(1+q,\nu),(1-q+\lambda,\mu-\nu),(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q},(1+\lambda,\mu)\end{matrix}\right.\right],\end{aligned}\tag{18}$$

for $\mu < \nu$,

$$
\begin{aligned}
&{}_zW_\infty^q\left\{x^\lambda H_{P,Q}^{M,N}\left[yx^{-\mu}(x-z)^\nu\left|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right.\right]\right\}\\
=&\frac{(-1)^q}{\Gamma(-q)}z^{\lambda-q}H_{P+1,Q+2}^{M+1,N+1}\left[yz^{-\mu+\nu}\left|\begin{matrix}(1+q,\nu),(a_j,e_j)_{1,P}\\(q-\lambda,\nu-\mu),(b_j,f_j)_{1,Q},(1+\lambda,\mu)\end{matrix}\right.\right],
\end{aligned}
\tag{19}
$$

and for $\mu = \nu$,

$$
\begin{aligned}
&{}_zW_\infty^q\left\{x^\lambda H_{P,Q}^{M,N}\left[yx^{\mu}(x-z)^{-\nu}\left|\begin{matrix}(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q}\end{matrix}\right.\right]\right\}\\
=&\frac{(-1)^q\Gamma(q-\lambda)}{\Gamma(-q)}z^{\lambda-q}H_{P+1,Q+2}^{M,N+1}\left[y\left|\begin{matrix}(1+q,\nu),(a_j,e_j)_{1,P}\\(b_j,f_j)_{1,Q},(1+\lambda,\mu)\end{matrix}\right.\right].
\end{aligned}
\tag{20}
$$

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