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Coefficient Estimates for Concave Partial Sums of Univalent Functions

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Abstract

In this paper, we consider the class (Σ_α), $\alpha \geq 0$, of analytic functions $f : U \to \overline{C}$, where U is an open unit disk, satisfying the standard normalization $f(0) = f'(0) - 1 = 0$ take the form

$$f(z) = z + \sum_{n=2}^{\infty} A_n(f) z^{n+\alpha}, \ \ \alpha \geq 0.$$

And assume that its partial sums $\sigma(f)$ satisfies the following conditions:
(i) $\sigma(f) \in \Sigma_\alpha$ is meromorphic in U and has a simple pole at the point p.
(ii) $\sigma(f)(0) = \sigma(f)'(0) - 1 = 0$.
(iii) $\sigma(f)$ maps U conformally onto a set whose complement with respect to $\overline{C}$ is convex.
We call such functions concave univalent functions. We prove some coefficient estimates for the partial sums $\sigma(f)$ such that

$$\sigma(f)(z) = \sum_{n=0}^{\infty} a_n (z-p)^{n+\alpha}, \ \ |z-p| < 1-p, \ p \in (0,1), \ \ \alpha \geq 0.$$

Moreover the pre-Schwarzian derivative of f are determined.

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1 Introduction and preliminaries.

In the theory of univalent functions the most interesting problem is to find the coefficient estimates for functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n \tag{1.1}$$

that are analytic and univalent in the unit disk $U := \{z : |z| < 1\}$. Let Co(p) be the family of functions $f : U \to \overline{C}$ where $p \in (0,1)$ that satisfy the following assumption

Assumption (A):
(i) f is meromorphic in U and has a simple pole at the point p.
(ii) $f(0) = f'(0) - 1 = 0$.
(iii) f maps U conformally onto a set whose complement with respect to $\overline{C}$ is convex.

The family Co(p) has been investigated recently in [1-4,18]. In [14], Livingston introduced a necessary and sufficient condition for a function f to be in Co(p)

$$\Re\{-(1+p^2) + 2pz - \frac{(z-p)(1-pz)f''(z)}{f'(z)}\} > 0, \ \ \forall\, z \in U.$$

In [18], Avkhadiev and Wirths proved that for each $f \in Co(p)$ with the expansion in (1.1), the inequality

$$|a_n(f) - \frac{1-p^{2n+2}}{p^{n-1}(1-p^4)}| \leq |\frac{p^2(1-p^{2n-2})}{p^{n-1}(1-p^4)}|$$

is valid. Equality is attained if and only if

$$f(z) = \frac{z - \frac{p}{1+p^2}(1+e^{i\theta})z^2}{(1-\frac{z}{p})(1-zp)}. \tag{1.2}$$

Finally, Bhowmik and Pommerenke [6] obtained certain coefficient estimates for functions have the Laurent expansion

$$f(z) = \sum_{n=-1}^{\infty} A_n(f)(z-p)^n, \ \ z \in \triangle$$

where $\triangle := \{z \in C : |z-p| < 1-p\}$ and $p \in (0,1)$,

$$|A_{n-2} - \frac{(1-p^2 A_{n-1})}{p}| \leq \frac{p}{(1-p^4)(1-p)^{n-1}}[1 - (\frac{1-p^4}{p^4})|A_{-1} + \frac{p^2}{1-p^4}|^2, \ \ n \geq 3$$

and of the form (1.2).

In this work, we consider the class (Σ_α), $\alpha > 0$, of analytic functions $f : U \to \overline{C}$, where U is an open unit disk, satisfying the standard normalization $f(0) = f'(0) - 1 = 0$ take the form

$$f(z) = z + \sum_{n=2}^{\infty} A_n(f) z^{n+\alpha}, \ \ \alpha \geq 0. \tag{1.3}$$

Recently, in [7] the authors studied the coefficient bounds for concave functions in the class (Σ_α). Assume that $f \in \Sigma_\alpha$ satisfies the assumption (A).

Now for the functions $f \in \Sigma_\alpha$ we define the following partial sums

$$s_k(z) = \sum_{n=2}^{k} a_n z^{n+\alpha}, \ \ z \in U,$$

such that $s_1(z) = z$, we construct the Cesáro means $\sigma_k f(z)$ of $f \in \Sigma_\alpha$ by

$$\begin{aligned}
\sigma_k f(z) &= \frac{1}{k}[s_1(z) + \sum_{n=2}^{k} s_n(z)] \\
&= \frac{1}{k}\big[s_1(z) + .. + s_k(z)\big] \\
&= \frac{1}{k}\big[z + (z + A_2 z^{\alpha+2}) + .. + (z + ... + A_k z^{\alpha+k})\big] \\
&= \frac{1}{k}\big[kz + (k-1)A_2 z^{\alpha+2} + .. + A_k z^{\alpha+k})\big] \\
&= z + \sum_{n=2}^{k} \big(\frac{k-n+1}{k}\big) A_n z^{\alpha+n} \\
&= f(z) * [z + \sum_{n=2}^{k} \big(\frac{k-n+1}{k}\big) z^{n+\alpha}] \\
&= f(z) * g_k(z)
\end{aligned} \tag{1.4}$$

where

$$g_k = z + \sum_{n=2}^{k} \big(\frac{k-n+1}{k}\big) z^{n+\alpha}.$$

Assume that $\sigma_k f(z)$ has the expansion

$$\sigma_k f(z) = \sum_{n=0}^{k} a_n(f)(z-p)^{n+\alpha}, \ \ \alpha \geq 0, \ z \in \triangle. \tag{1.5}$$

and satisfies the assumption (A).

Note that the classical Cesáro means play an important role in geometric function theory (see [8, 15-17]).

We determined the coefficient bounds for the partial sums (1.4) of the functions $f \in \Sigma_\alpha$ and discuss some other properties containing (1.4) such as the per-Schwarzian derivative.
For this purpose, we need to the following results

Theorem 1.1. [18] For each $f \in Co(p)$, there exists a function ω holomorphic in U such that $\omega(U) \subset \overline{U}$ and

$$f(z) = \frac{z - \frac{p}{1+p^2}(1+\omega(z))z^2}{(1-\frac{z}{p})(1-zp)}, \quad z \in U. \tag{1.6}$$

Lemma 1.2. [5] Let $q \in (1,2]$ and $f \in Co(q)$. Then there exists a function ω holomorphic in U such that $\omega(U) \subset \overline{U}$ and

$$f'(z) = \frac{(1+z\omega(z))^{q-1}}{(1-z)^{q+1}} \quad z \in U.$$

2 Coefficient bounds.

In this section, we introduce some coefficient estimates for functions belong to the class Σ_α and have the expansion (1.4). Now, we state our first result

Theorem 2.1. Let $p \in (0,1)$ and $\sigma_k f \in Co(p)$ has the expansion (1.5) where $f \in \Sigma_\alpha$. Then

$$|a_0| \leq \frac{1}{p(1+p^2)}. \tag{2.1}$$

The inequality is sharp.

Proof. Let $\sigma_k f \in Co(p)$. Then by Theorem 1.1, there exists a function $\omega(z)$ holomorphic in U and $\omega(U) \subset \overline{U}$ satisfying (1.6). Assume that

$$\omega(z) = \sum_{n=0}^{\infty} c_n (z-p)^{n+\alpha}, \quad z \in \triangle. \tag{2.2}$$

Using these two expansions (1.5) and (2.2), the power series formulation of (1.6) takes the form

$$\sum_{n=0}^{k} a_n (z-p)^{n+\alpha}[\frac{1}{z} - 2p + zp^2] = 1 - \frac{p}{1+p^2}[1 + \sum_{n=0}^{\infty} c_n (z-p)^{n+\alpha}]z. \tag{2.3}$$

Comparing the coefficient of $z(z-p)^{\alpha}$ on both sides of (2.3) and using the inequality $|c_0| < 1$, yields the assertion (2.1).

Theorem 2.2. Let $p \in (0,1)$ and $\sigma_k f \in Co(p)$ has the expansion (1.5) where $f \in \Sigma_{\alpha}$. Then

$$|a_n(f)| \leq \frac{1}{p(1+p^2)(\alpha+1)_n}, \quad n \geq 2. \tag{2.4}$$

where the symbol $(\alpha+1)_n$ is a Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n = 0 \\ a(a+1)...(a+n-1), & n = \{1,2,...\}. \end{cases}$$

The inequality is sharp.

Proof. Let $\sigma_k f \in Co(p)$. Then by Comparing the coefficient of $z(z-p)^{n+\alpha}$ on both sides of (2.3) and using the inequality (see [8])

$$|c_n| \leq \frac{1}{(\alpha+1)_n},$$

yields the assertion (2.4).

Consequently, the next result present sharp coefficient estimates for all $n \geq 2$ if $\sigma_k f \in Co(p)$ of the form (1.4) and has the expansion (1.5).

Theorem 2.3. Let $p \in (0,1)$ and $\sigma_k f \in Co(p)$ of the form (1.4) and have the expansion (1.5). Then

$$|A_n(f)| \leq \frac{k}{p(k-n+1)(1+p^2)(\alpha+1)_n}, \quad n \geq 2, \quad k = 1,2,..., \tag{2.5}$$

where $\alpha \geq 0$. The inequality is sharp.

Proof. Equating the right sides of (1.4) and (1.5) and applying Theorems 2.1.

Theorem 2.4. Let $q \in (1,2]$ and $\sigma_k f \in Co(q)$ where $f \in \Sigma_{\alpha}$. Then

$$|a_n(f)| \leq \frac{1}{3(n+\alpha)(\alpha+1)_n}, \quad n \geq 2. \tag{2.6}$$

Proof. Let $\sigma_k f \in Co(q)$. Then by Lemma 1.1, there exists a function $\omega(z)$ holomorphic in U and $\omega(U) \subset \overline{U}$ satisfying

$$(\sigma_k f)' = \frac{(1+z\omega(z))^{q-1}}{(1-z)^{q+1}} \quad z \in U, \tag{2.7}$$

where

$$\sigma_k f(z) = \sum_{n=0}^{k} a_n(f)(z-q)^{n+\alpha}, \ \ \alpha \geq 0, \ z \in \triangle_q, \tag{2.8}$$

$$\omega(z) = \sum_{n=0}^{\infty} d_n(z-q)^{n+\alpha-1}, \ \ z \in \triangle_q \tag{2.9}$$

and

$$\triangle_q := \{z \in C : |z-q| < 2-q, \ \ q \in (1,2]\}.$$

Now substitute (2.9) in (2.7) and using (2.8), then comparing the coefficient of $z(z-p)^{n+\alpha-1}$ on both sides and using the inequality

$$|d_n| \leq \frac{1}{(\alpha+1)_n},$$

yields the assertion (2.6).

Theorem 2.5. Let $q \in (1,2]$ and $\sigma_k f \in Co(q)$ where $f \in \Sigma_\alpha$. Then

$$|A_n(f)| \leq \frac{k}{3(k-n+1)(n+\alpha)(\alpha+1)_n}, \ \ n \geq 2, \ \ k = 1,2,..., \tag{2.10}$$

where $\alpha \geq 0$. The inequality is sharp.

Proof. Equating the right sides of (1.4) and (2.8) and applying Theorem 2.4, we get the required result.

3 Norm estimates of the pre-Schwarzian derivative

In this section, we discuss some other properties of the operator (1.3) such as the univalence of this operator by using pre-Schwarzian derivative.
Let h be analytic and locally univalent in U. The pre-Schwarzian derivative T_h of $h \in \Sigma_\alpha, \ \alpha \geq 0$ is defined by

$$T_h(z) = \frac{h''(z)}{h'(z)}, \ \ (z \in U) \tag{3.1}$$

with the norm

$$\|T_h\| = sup_{z \in U}|T_h|(1-|z|^{2+\alpha}).$$

It is known that $\|T_h\| < \infty$ if and only if h is uniformly locally univalent. It is also known that $\|T_h\| \leq 6$ for $h \in \mathcal{S}$ the class of starlike functions and that $\|T_h\| \leq 4$ for $h \in \mathcal{K}$ the class of convex functions (see [13]).

Theorem 3.1. Let $p \in (0,1)$ and $\sigma_k f \in Co(p)$ of the form (1.4). Then for $z \to 0$ the pre-Schwarzian derivative of $\sigma_k f$ satisfies the inequality

$$\|T_{\sigma_k f}\| \leq \frac{(p+2)^2(1+p)}{p(1+p^2)}.$$

The result is sharp.

Proof. Let $p \in (0,1)$ and $\sigma_k f \in Co(p)$ then in view of Theorem 1.1, $\sigma_k f$ takes the form (1.6). Differentiating both sides of (1.6) we obtain

$$\sigma_k f'(z) = \frac{H(z)(1-W'(z)) - (z-W(z))H'(z)}{H^2(z)}$$

where $H(z) := (1-\frac{z}{p})(1-zp)$ and $W(z) := \frac{p}{1+p^2}(1+\omega(z))z^2$, or equivalent to

$$\ln \sigma_k f'(z) = \ln[H(z)(1-W'(z)) - (z-W(z))H'(z)] - 2\ln H(z).$$

Take the derivative for the above equality we receive

$$\frac{\sigma_k f''(z)}{\sigma_k f'(z)} = \frac{Q'(z)}{Q(z)} - \frac{2H'(z)}{H(z)}$$

where

$$Q(z) := [H(z)(1-W'(z)) - (z-W(z))H'(z)].$$

Now for $z \to 0$ we obtain the assertion (2.10).

Corollary 3.2. Let $p \in (0,1)$ and $\sigma_k f \in Co(p)$. Then $\sigma_k f$ is uniformly locally univalent when $z \to 0$.

Proof. By applying Theorem 3.1 we get $\|T_{\sigma_k f}\| < \infty$ hence $\sigma_k f$ is uniformly locally univalent.

Theorem 3.3. Let $q \in (1,2]$ and $\sigma_k f \in Co(q)$ where $f \in \Sigma_\alpha$. Then for $z \to 0$ the pre-Schwarzian derivative of $\sigma_k f$ satisfies the inequality

$$\|T_{\sigma_k f}\| \leq 4.$$

The result is sharp.

Proof. Let $q \in (1,2]$ and $\sigma_k f \in Co(q)$ where $f \in \Sigma_\alpha$. Now when $q = 2$, (2.7) is equivalent to

$$\ln(\sigma_k f)' = \ln(1 + z\omega(z)) - 3\ln(1 - z) \;\; z \in U,$$

take the derivative we obtain

$$\frac{(\sigma_k f)''}{(\sigma_k f)'} = \frac{z\omega'(z) + \omega(z)}{(1 + z\omega(z))} + \frac{3}{(1 - z)}.$$

Since Lemma 1.1 holds when $\omega(z) := e^z$ (see [13]) then for $z \to 0$ we have

$$\|T_{\sigma_k f}\| < 4.$$

Corollary 3.4. Let the assumption of Theorem 3.3 holds. Then $\sigma_k f$ is uniformly locally univalent when $z \to 0$.

Proof. By applying Theorem 3.3 we get $\|T_{\sigma_k f}\| < \infty$ hence $\sigma_k f$ is uniformly locally univalent.

Corollary 3.5. Let the assumption of Theorem 3.3 holds. Then $\sigma_k f$ is convex as $z \to 0$.

Proof. By applying Theorem 3.3 we get $\|T_{\sigma_k f}\| < 4$ hence $\sigma_k f$ is convex.

Note that for other work related to partial sums and also on Cesáro can be found in (see for examples [9-12]).

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References

[1] F. G. Avkhadiev and K. J. Wirths, Convex holes produce lower bounds for coefficients, *Complex Variables*, 47(2002),553-563.

[2] F. G. Avkhadiev, C. Pommerenke and K. J. Wirths, On the sufficient of concave univalent functions, *Math. Nachr.* 271 (2004),3-9.

[3] F. G. Avkhadiev and K. J. Wirths, On a conjecture of Livingston, *Mathematica(Cluj)*, 46(69)(2004), 19-23.

[4] F. G. Avkhadiev and K. J. Wirths, A proof of Livingston conjecture, *Forum Math.* 19(2007), 149-158.

[5] F. G. Avkhadiev and K. J. Wirths, Concave Schlicht functions with bounded opening angle at infinity, *Lobachevskii Journal of Mathematics*, 17 (2005), 3-10.

[6] B. Bhowmik and C. Pommerenke, Coefficient inequalities for concave and meromorphically starlike univalent functions, *Ann. Polon. Math.* 93.2(2008), 177-186.

[7] M. Darus and R. W. Ibrahim, Coefficient inequalities for a new class of univalent functions, *Lobachevskii Journal of Mathematics*, 29(4) (2008), 221-229.

[8] M. Darus and R. W. Ibrahim, On Cesáro means for Fox-Wright functions, *J. Math. and Stat.* 4(3) (2008), 156-160.

[9] R. W. Ibrahim and M. Darus, Partial sums for certain classes of meromorphic functions, *Tamkang Journal of Mathematics*. 41(1) (2010), 39-49.

[10] M. Darus and R. W. Ibrahim, On Cesáro means of order μ for outer functions, *Inter. J. Nonlinear Science*, 9(4) (2010), 455-460.

[11] M. Darus and R. W. Ibrahim, Coefficient inequalities for concave Cesáro operator of non-concave analytic functions, *European J. Pure and Appl. Math.(EJPAM)*, 3(6) (2010), 1086 1092.

[12] M. Darus and R. W. Ibrahim, On Cesáro means of hypergeometric functions, *Studii si Cercetari Stiintifice. Seria Matematica*, 19(1) (2009), 59-72.

[13] Y. C. Kim and T. Sugawa, Norm estimates of the per-Schwarzian derivative for certain classes of univalent functions, *Proc. of the Edinburg Mathematical Society*, 49(2006), 131-143.

[14] A. E. Livingston, Convex meromorphic mappings, *Ann. Polon. Math.* 59(1994), 275-291.

[15] S. Ruscheweyh, *Convolutions in Geometric Function Theory*, , Sem. Math. Sup., University of Montereal Press 1982.

[16] S. Ruscheweyh, Geometric properties of Cesáro means, Results Math.22(1992), 739-748.

[17] S. Ruscheweyh and L.Salinas, Subordination by Cesáro means, *Complex Var. Theory Appl.* 21(1993), 279-285.

[18] K. J. Wirths, On the residuum of cocave univalent functions, *Serdica Math. J.* 32(2006), 209-214.

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