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Weighted Composition Operators on Sobolev - Lorentz Spaces

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Abstract

For Ω an open subset of the Euclidean space $\mathbf{R}^n$, $T : \Omega \to \Omega$ a measurable non-singular transformation and u a real-valued measurable function on $\mathbf{R}^n$, we study boundedness of the weighted composition operator $uC_T : f \mapsto u \cdot (f \circ T)$ on the Sobolev-Lorentz space $W^{1,n,q}(\Omega)$, consisting of those functions of the Lorentz space $L(n, q)$, whose distributional derivatives of the first order belong to $L(n, q), 1 \leq q \leq n$.

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1 Introduction

Suppose $(\Omega, \mathcal{A}, \mu)$ is a measure space where Ω is an open subset of the Euclidean space $\mathbf{R}^n$, $\mathcal{A}$ the σ-algebra of Lebesgue measurable subsets of Ω and μ the Lebesgue measure. Let f be a complex-valued Lebesgue measurable function defined on Ω. For $s \geq 0$, define μ_f the distribution function of f as

$$\mu_f(s) = \mu\{x \in \Omega : |f(x)| > s\}.$$

By f^* we mean the non-increasing rearrangement of f given as

$$f^*(t) = \inf\{s > 0 : \mu_f(s) \leq t\}, \quad t \geq 0.$$

For $1 \leq q \leq n$, the Lorentz norm of f is given by

$$\|f\|_{n,q} = \left(\int_0^\infty (t^{1/n} f^*(t))^q \ \frac{dt}{t} \right)^{1/q}.$$

The Lorentz space $L(n, q)$ is the set of equivalence classes of complex-valued Lebesgue measurable functions f on Ω with $\|f\|_{n,q} < \infty$. $L(n, q)$ is a Banach space [11] with respect to above norm.

The *Sobolev-Lorentz space* $W^{1,n,q}(\Omega)$ is defined as the set of all complex-valued functions f in $L(n, q)$ whose weak partial derivatives $\partial f/\partial x_i$ (in the distributional sense) belong to $L(n, q)$, $i = 1, 2, \ldots, n$. It is a Banach space [25] with respect to the norm:

$$\|f\|_{1,n,q} = \|f\|_{n,q} + \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{n,q}.$$

On σ-finite measure space $(\Omega, \mathcal{A}, \mu)$, let $T : \Omega \to \Omega$ be a *measurable* ($T^{-1}(E) \in \mathcal{A}$ for every $E \in \mathcal{A}$) *non-singular transformation* ($\mu(T^{-1}(E))$=0, whenever $\mu(E) = 0$). Let the function $f_T = d(\mu \circ T^{-1})/d\mu$ be the Radon-Nikodym derivative. Suppose u is a complex-valued measurable function defined on $\mathbf{R}^n$. Then T induces a well-defined linear transformation uC_T on $W^{1,n,q}(\Omega)$ defined by

$$(uC_T f)(x) = u(x) f(T(x)),\ x \in \Omega,\ f \in W^{1,n,q}(\Omega).$$

If uC_T maps $W^{1,n,q}(\Omega)$ into itself and is bounded, then we call uC_T a *weighted composition operator* on $W^{1,n,q}(\Omega)$ induced by T with weight u. If $u \equiv 1$, then C_T is called a *composition operator* induced by T.

Our study here, of weighted composition operators, on Sobolev-Lorentz space $W^{1,n,q}(\Omega)$ is motivated by the work of *Herbert Kamowitz* and *Dennis Wortman*[12]. Other similar references include [2–7, and 15–17]. The paper contains two sections. In the first section we define the composition operator on $W^{1,n,q}(\Omega)$, and the second section is devoted to the study of weighted composition operator on $W^{1,n,q}(\Omega)$.

2 Composition operator on $W^{1,n,q}(\Omega)$

Lemma 2.1 *Let $f_T, \frac{\partial T_k}{\partial x_i} \in L^\infty(\mu)$ with $\left\| \frac{\partial T_k}{\partial x_i} \right\|_\infty \leq M, (M > 0)\ i, k = 1, 2, \ldots, n$, where $T = (T_1, T_2, \ldots, T_n)$ and $\frac{\partial T_k}{\partial x_i}$ denotes the first order partial derivative(in the classical sense). Then for each f in $W^{1,n,q}(\Omega)$ we have*

$f \circ T \in L(n,q), 1 \leq q \leq n$, and the first order distributional derivatives of $(f \circ T)$, given by,

$$\frac{\partial}{\partial x_i}(f \circ T) = \sum_{k=1}^{n}\Big(\frac{\partial f}{\partial x_k} \circ T\Big)\frac{\partial T_k}{\partial x_i} \tag{1}$$

for $i = 1, 2, \ldots, n$, are in $L(n,q)$.

Proof. The Radon-Nikodym derivative $f_T = d(\mu \circ T^{-1})/d\mu \in L^\infty(\mu)$ implies that for each $E \in \mathcal{A}$,

$$(\mu \circ T^{-1})(E) = \int_E f_T d\mu \ \leq \ \|f_T\|_\infty \mu(E).$$

For f in $W^{1,n,q}(\Omega)$, the distribution of $f \circ T$ satisfies

$$\begin{aligned} \mu_{f\circ T}(s) &= \mu\{x \in \Omega : |f(T(x))| > s\} \\ &= (\mu \circ T^{-1})\{x \in \Omega : |f(x)| > s\} \\ &\leq \|f_T\|_\infty \ \mu\{x \in \Omega : |f(x)| > s\} = \|f_T\|_\infty \mu_f(s). \end{aligned}$$

Therefore

$$\{s > 0 : \mu_f(s) \leq t\} \subseteq \{s > 0 : \mu_{f\circ T}(s) \leq \|f_T\|_\infty t\}.$$

This gives

$$(f \circ T)^*(\|f_T\|_\infty t) \leq f^*(t), \text{ i.e., } (f \circ T)^*(t) \leq f^*(t/\|f_T\|_\infty).$$

Now $f \in L(n,q), 1 \leq q \leq n$, gives

$$\begin{aligned} \|f \circ T\|_{n,q}^q &= \int_0^\infty (t^{1/n}(f \circ T)^*(t))^q \frac{dt}{t} \\ &\leq \int_0^\infty (t^{1/n} f^*(t/b))^q \frac{dt}{t} = \|f_T\|_\infty^{q/n} \|f\|_{n,q}^q. \end{aligned}$$

Thus $\|f \circ T\|_{n,q} \ \leq \ \|f_T\|_\infty^{1/n} \|f\|_{n,q}$, and hence $f \circ T \in L(n,q)$.

By the same arguments, as each weak partial derivative $\partial f/\partial x_k \in L(n,q)$, it follows that $\partial f/\partial x_k \circ T \in L(n,q)$, for each $k = 1, 2, \ldots, n$.
Also $\partial T_k/\partial x_i \in L^\infty(\mu)$, therefore

$$\Big(\frac{\partial f}{\partial x_k} \circ T\Big)\frac{\partial T_k}{\partial x_i} \in L(n,q), \quad \text{for each } i, k = 1, 2, \ldots, n.$$

Hence, using triangle inequality, it follows that the function in right hand side of (1) belongs to $L(n,q)$, for each $i = 1, 2, \ldots, n$.

Since $f \in W^{1,n,q}(\Omega), 1 \leq q \leq n$, following the same computation as in Friedrich's Theorem [14, Theorem 2.2.1, p. 57], there exists a sequence $< f_m >$

in $\mathcal{D}(\mathbf{R}^n) = C_0^\infty(\mathbf{R}^n)$ such that $f_m \to f$ in $L(n,q)(\Omega)$ and $\frac{\partial f_m}{\partial x_i} \to \frac{\partial f}{\partial x_i}$ in $L(n,q)(\Omega^{'})$ for every $1 \le i \le n$ and for every relatively compact set $\Omega^{'}$ in Ω

Let $g \in \mathcal{D}(\mathbf{R}^n)$. We choose relatively compact set $\Omega^{'}$ in Ω with $\text{supp}(g) \subset \Omega^{'}$. Then by the ordinary chain rule for smooth function f_m, we have for each $i = 1, 2, \ldots, n$

$$\begin{aligned}\int_\Omega (f_m \circ T)\frac{\partial g}{\partial x_i} d\mu &= \int_{\Omega'} (f_m \circ T)\frac{\partial g}{\partial x_i} d\mu \\ &= -\int_{\Omega'} \frac{\partial}{\partial x_i}(f_m \circ T) g d\mu \\ &= -\int_{\Omega'} \sum_{k=1}^n \Big(\frac{\partial f_m}{\partial x_k} \circ T\Big)\frac{\partial T_k}{\partial x_i} g d\mu. \end{aligned} \qquad (2)$$

Now $g \in \mathcal{D}(\mathbf{R}^n)$ implies that for each i, g, $\dfrac{\partial g}{\partial x_i} \in X^{'}$, where $X^{'}$ is the associate space of the Banach function space $X = L(n,q)(\Omega),\ 1 \le q \le n$. Therefore by using the *Hölder's inequality* in Banach function spaces, we have

$$\begin{aligned}\int_\Omega |\, f_m \circ T - f \circ T \,| \left|\frac{\partial g}{\partial x_i}\right| d\mu &\le \|(f_m - f)\circ T\|_X \left\|\frac{\partial g}{\partial x_i}\right\|_{X'} \\ &\le \|f_T\|_\infty^{1/n} \|f_m - f\|_X \left\|\frac{\partial g}{\partial x_i}\right\|_{X'} \to 0.\end{aligned}$$

Therefore as $m \to \infty$, for each $i = 1, 2, \ldots, n$

$$\int_\Omega (f_m \circ T)\frac{\partial g}{\partial x_i} d\mu \to \int_\Omega (f \circ T)\frac{\partial g}{\partial x_i} d\mu.$$

By the similar arguments, using $\frac{\partial f_m}{\partial x_k} \to \frac{\partial f}{\partial x_k}$ in $L(n,q)(\Omega^{'})$ and $\frac{\partial T_k}{\partial x_i} \in L^\infty(\mu)$, we obtain that in $L(n,q)(\Omega^{'})$,

$$\sum_{k=1}^n \Big(\frac{\partial f_m}{\partial x_k} \circ T\Big)\frac{\partial T_k}{\partial x_i} \to \sum_{k=1}^n \Big(\frac{\partial f}{\partial x_k} \circ T\Big)\frac{\partial T_k}{\partial x_i}$$

So by the *Hölder's inequality* in Banach function spaces again, we obtain as $m \to \infty$, for each $i = 1, 2, \ldots, n$

$$\int_{\Omega'} \sum_{k=1}^n \Big(\frac{\partial f_m}{\partial x_k} \circ T\Big)\frac{\partial T_k}{\partial x_i} g d\mu \to \int_{\Omega'} \sum_{k=1}^n \Big(\frac{\partial f}{\partial x_k} \circ T\Big)\frac{\partial T_k}{\partial x_i} g d\mu.$$

Hence by taking limits on both the sides of (2) as $m \to \infty$, we obtain

$$\begin{aligned}\int_\Omega (f \circ T)\frac{\partial g}{\partial x_i} d\mu &= -\int_{\Omega'} \sum_{k=1}^n \Big(\frac{\partial f}{\partial x_k} \circ T\Big)\frac{\partial T_k}{\partial x_i} g d\mu \\ &= -\int_{\Omega} \sum_{k=1}^n \Big(\frac{\partial f}{\partial x_k} \circ T\Big)\frac{\partial T_k}{\partial x_i} g d\mu.\end{aligned}$$

Therefore for all $i = 1, 2, \ldots, n$

$$-\int_\Omega \frac{\partial}{\partial x_i}(f \circ T) g d\mu = -\int_\Omega \sum_{k=1}^{n} \Big(\frac{\partial f}{\partial x_k} \circ T\Big) \frac{\partial T_k}{\partial x_i} g d\mu.$$

As g was chosen arbitrarily, the equation (1) follows.

Theorem 2.2 *Let $\Omega \subset \mathbf{R}^n$ be an open set and $T : \Omega \to \Omega$ a measurable non-singular transformation with Radon-Nikodym derivative $f_T = d(\mu \circ T^{-1})/d\mu$, $\frac{\partial T_k}{\partial x_i}$ in $L^\infty(\mu)$, and $\Big\|\frac{\partial T_k}{\partial x_i}\Big\|_\infty \leq M, M > 0$, for $i, k = 1, 2, \ldots, n$, where $T = (T_1, T_2, \ldots, T_n)$ and $\frac{\partial T_k}{\partial x_i}$ denotes the first order partial derivatives (in the classical sense). Then the mapping C_T defined by $C_T(f) = f \circ T$ is a composition operator on the Sobolev-Lorentz space $W^{1,n,q}(\Omega)$, $1 \leq q \leq n$.*

Proof. By the Lemma 2.1, we have $f \circ T \in W^{1,n,q}(\Omega)$ and its norm satisfies the following:

$$\begin{aligned}
\|f \circ T\|_{1,n,q} &= \|f \circ T\|_{n,q} + \sum_{i=1}^{n} \Big\|\frac{\partial}{\partial x_i}(f \circ T)\Big\|_{n,q} \\
&= \|f \circ T\|_{n,q} + \sum_{i=1}^{n} \Big\|\sum_{k=1}^{n} \Big(\frac{\partial f}{\partial x_k} \circ T\Big)\frac{\partial T_k}{\partial x_i}\Big\|_{n,q} \\
&\leq \|f_T\|_\infty^{1/n} \, \|f\|_{n,q} + \sum_{i=1}^{n}\sum_{k=1}^{n} \|f_T\|_\infty^{1/n} \, \Big\|\frac{\partial f}{\partial x_k}\Big\|_{n,q} \, \Big\|\frac{\partial T_k}{\partial x_i}\Big\|_\infty \\
&\leq \|f_T\|_\infty^{1/n} \, \|f\|_{n,q} + \|f_T\|_\infty^{1/n} M n \sum_{k=1}^{n} \Big\|\frac{\partial f}{\partial x_k}\Big\|_{n,q} \\
&\leq \|f_T\|_\infty^{1/n}(1 + nM)\|f\|_{1,n,q}
\end{aligned}$$

The result follows.

3 Weighted composition operator on $W^{1,n,q}(\Omega)$

Suppose u is a real-valued measurable function defined on $\mathbf{R}^n$. Also suppose that $T : \Omega \to \Omega$ is a measurable non-singular transformation and $(\Omega, \mathcal{A}, \mu)$ is the σ-finite measure space, where Ω an open subset of $\mathbf{R}^n$. On the same lines as in Lemma 2.1, we have the following, for $i = 1, 2, \ldots, n$.

$$\frac{\partial}{\partial x_i}(u \cdot (f \circ T)) = \frac{\partial u}{\partial x_i}(f \circ T) + u \sum_{k=1}^{n} \Big(\frac{\partial f}{\partial x_k} \circ T\Big) \frac{\partial T_k}{\partial x_i}.$$

Theorem 3.1 *If all the conditions stated in the Theorem 2.2 are satisfied and, in addition,* $u \in L^\infty(\mu)$ *such that the first order classical partial derivatives* $\partial u/\partial x_i$ *satisfy* $\|\partial u/\partial x_i\|_\infty \le M_1, M_1 > 0$*, for* $i = 1, 2, \ldots, n$*, then the mapping* uC_T *defined by* $(uC_T)f = u\cdot(f \circ T)$ *is a weighted composition operator on the Sobolev-Lorentz space* $W^{1,n,q}(\Omega),\ 1 \le q \le n$.

Proof. By the same arguments as in Lemma 2.1, we find

$$\begin{aligned} \|u\cdot(f\circ T)\|_{n,q} &\le \|u\|_\infty \|f_T\|_\infty^{1/n} \|f\|_{n,q}, \\ \left\|\frac{\partial u}{\partial x_i}(f\circ T)\right\|_{n,q} &\le M_1 \|f_T\|_\infty^{1/n} \|f\|_{n,q}, \end{aligned}$$

and

$$\left\|u\sum_{k=1}^{n}\Big(\frac{\partial f}{\partial x_k}\circ T\Big)\frac{\partial T_k}{\partial x_i}\right\|_{n,q} \le \|u\|_\infty\ M\|f_T\|_\infty^{1/n}\sum_{k=1}^{n}\left\|\frac{\partial f}{\partial x_k}\right\|_{n,q}.$$

Hence it follows that

$$\begin{aligned} &\|(uC_T)f\|_{1,n,q} \\ &= \|u\cdot(f\circ T)\|_{1,n,q} = \|u\cdot(f\circ T)\|_{n,q} + \sum_{i=1}^{n}\left\|\frac{\partial}{\partial x_i}(u\cdot(f\circ T))\right\|_{n,q} \\ &\le \|u\|_\infty\|f_T\|_\infty^{1/n}\|f\|_{n,q} + \sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}(f\circ T)\right\|_{n,q} + \sum_{i=1}^{n}\left\|u\sum_{k=1}^{n}\Big(\frac{\partial f}{\partial x_k}\circ T\Big)\frac{\partial T_k}{\partial x_i}\right\|_{n,q} \\ &\le \|u\|_\infty\|f_T\|_\infty\|f\|_{1,n,q} + nM_1\|f_T\|_\infty^{1/n}\|f\|_{1,n,q} + nM\|u\|_\infty\|f_T\|_\infty^{1/n}\|f\|_{1,n,q} \end{aligned}$$

Thus $\|(uC_T)f\|_{1,n,q} \ \le\ K\|f\|_{1,n,q}$, for some $K > 0$.

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