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On LA-Modules

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Abstract. In this study we promoted the notion of LA-module over an LA-ring defined in [7] and further established the substructures, operations on substructures and quotient of an LA-module by its LA-submodule. We also indicated the non similarity of an LA-module to the usual notion of a module over a commutative ring.

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1. Introduction

M. Kazim and M. Naseeruddin in [2] introduced the concept of left almost semigroup. A groupoid S is called a left almost semigroup (abbreviated as an LA-semigroup) if its elements satisfy the left invertive law:

$$(ab)\,c = (cb)\,a \text{ for all } a, b, c \in S$$

It is also called an Abel-Grassmann's groupoid (abbreviated as an AG-groupoid) [6]. LA-semigroup generalizes commutative semigroup. Later, the structure was explored in [3] by Q. Mushtaq. Further Q. Mushtaq and S. Kamran in [5] extend the notion to left almost groups. Properties of left almost groups(abbreviated as LA-group) are close to commutative groups. It is interesting to note that, we can factor an LA-group by any of its LA-subgroup. We know that if G is a group and H is its subgroup, then $H\,(ab) \neq (Ha)\,(Hb)$, unless H is normal in G. Here in case of an LA-group, there is no such restriction [5].

Due to non smooth structural formation, many authors established several useful results for LA-semigroups and LA-groups. With these inspirations in [8], the concept of LA-ring is initiated. A non-empty set R with two binary operations "+" and "·" is called a left almost ring, if $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup and left, right distributive laws of "·" over "+" hold.

In continiution, T. Shah and Inayatur Rehman in [7] utilized the both LA-semigroup and LA-ring and generalizes the notion of a commutative semigroup ring. Further T. Shah and Inayatur Rehman define the notion of LA-module over an LA-ring, a non abelian non associative structure (cf. [7, Page 214]) but closer to abelian group. So study of this structure is completely parallel to the modules which are basically the abelian groups.
In this study, we investigate elementary properties, substructures, operations on substructures and quotient of an LA-module by its LA-submodule. We, also indicated the non similarity of an LA-module to the usual notion of a module.

2. LA-Module

In this section, we use the definition of an LA-module over an LA-ring. We study some examples of LA-module and also discuss the elementary properties, substructure and operations on substructures of an LA-module.

Definition 1. [7, Page 214] Let $(R, +, \cdot)$ be an LA-ring with left identity 1. An LA-group $(M, +)$ is called an LA-module over R, if the map $R \times M \longrightarrow M$ is defined as $(r, m) \longmapsto rm \in M$, where $r \in R$ and $m \in M$ satisfies:
(LAM1) $r\,(m_1 + m_2) = rm_1 + rm_2$
(LAM2) $(r_1 + r_2)\,m = r_1 m + r_2 m$
(LAM3) $r_1\,(r_2 m) = r_2\,(r_1 m)$
(LAM4) $1 \cdot m = m$, for all $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$.

Remark 1. View the above axiom (LAM3) which represent the difference between LA-module and module.

Example 1. Every LA-ring with left identity is an LA-module over itself.

Example 2. [7, Page 214] Let $(R,+,\cdot)$ be an LA-ring with left identity and S be a commutative semigroup. Then $R[S] = \{\sum_{j=1}^{n} a_j s_j : a_j \in R, s_j \in S\}$ is an LA-module over R.

Lemma 1. [4, lemma 1 & 2] *If $(G, .)$ is a locally associative LA-semigroup, then*
(1) $a^m a^n = a^{m+n}$, (2) $(a^m)^n = a^{mn}$, for all $m, n \in \mathbb{Z}^+$

Proposition 1. *Every locally associative LA-group is an LA-module over the ring of integers.*

Proof. Let M be a locally associative LA-group. Define a map $\mathbb{Z} \times M \longrightarrow M$ by $(n, m) \mapsto nm$.Let $n, n_1, n_2 \in \mathbb{Z}$ and $m, m_1, m_2 \in M$.
(LAM1): $n\,(m_1 + m_2) = nm_1 + rm_2$
We show this by induction. For this, let $n = 2$, $2\,(m_1 + m_2) = (m_1 + m_2) + (m_1 + m_2) = (m_1 + m_1) + (m_2 + m_2) = 2m_1 + 2m_2$, by locally associative property.

Suppose it is true for $n = k$, i.e $k(m_1 + m_2) = km_1 + km_2$.......(1)
Consider,

$$\begin{aligned}(k+1)(m_1+m_2) &= k(m_1+m_2)+(m_1+m_2), \text{ by locally associative property.}\\ &= (km_1+km_2)+(m_1+m_2), \text{ by (1)}\\ &= (km_1+m_1)+(km_2+m_2)\\ &= (k+1)m_1+(k+1)m_2, \text{ by locally associative property.}\end{aligned}$$

If $n = -k$, where k is any positive integer, then
(LAM2): $(n_1 + n_2)m = n_1m + n_2m$, by locally associativity.
(LAM3):

$$\begin{aligned}n_1(n_2m) &= n_1(m+m+...+m)_{n_2\text{-times}} = n_1m+n_1m+...+n_1m, \ n_2\text{-times}\\ &= n_2(n_1m).\end{aligned}$$

(LAM4): $1m = m$.
Hence M is an LA-module over the ring of integers $\mathbb{Z}$. ∎

2.1. **Elementary Properties of an LA-module.**

Theorem 1. *Let* $(M, +)$ *be an LA-module over an LA-ring* $(R, +, \cdot)$, *then*
(1) $r0_M = 0_M$.
(2) $0_Ra = 0_M$.
(3) $(-r)a = -ra = r(-a)$.
(4) $(-r)(-a) = ra$ *for all* $r, s \in R$ *and* $a, b \in M$.

Proof. (1) Consider $r0_M = r(0_M + 0_M) = r0_M + r0_M$ implies that $0_M + r0_M = r0_M + r0_M$. So, $0_M = r0_M$, because LA-group is cancellative.

(2) Consider $0_Ra = (0_R + 0_R)a = 0_Ra + 0_Ra$, implies that $0_M + 0_Ra = 0_Ra + 0_Ra$, so $0_Ra = 0_M$, because LA-group is cancellative. Similarly, one can easily verify (3) and (4). ∎

Corollary 1. *Let* $(R, +, \cdot)$ *be an LA-ring, then for all* $a, b, c \in R$
(1) $0 \cdot a = 0 = a \cdot 0$.
(2) $a(-b) = -ab = (-a)b$.
(3) $-(-a) = a$.
(4) $(-a)(-b) = ab$.

2.2. **LA-submodule.** In the spirit of a definition of a submodule of a module over a commutative ring, we initiate the following definition.

Definition 2. An LA-subgroup N of an LA-module M over an LA-ring R is called an LA-submodule over R, if $RN \subseteq N$, i.e., $rn \in N$ for all $r \in R$ and $n \in N$.

Remark 2. M and (0) are improper LA-submodules of LA-module M over LA-ring R.

Remark 3. Since an LA-ring $(R, +, \cdot)$ with left identity is an LA-module over itself. Therefor an ideal I of R is an LA-submodule of LA-module R, where the action of R on I is defined by $(r, i) \mapsto ri$ for all $r \in R$ and $i \in I$.

Example 3. Let M be an LA-module over an LA-ring R. Then for each $0 \neq r \in R$, $rM = \{rm : m \in M\}$ is an LA-submodule of M.

Remark 4. If R is an LA-ring and a is any non zero element of R, then aR, and Ra are left ideals of R.

Theorem 2. *If A and B are two LA-submodules of an LA-module M over an LA-ring R, then $A \cap B$ is also an LA-submodule of M.*

Corollary 2. *Intersection of any number of ideals of an LA-ring R is again an ideal of R.*

The following is the particular case of theorem 2.

Theorem 3. *If A and B are two ideals of an LA-ring R, then $A \cap B$ is also an ideal of an LA-ring R.*

Corollary 3. *Intersection of any number of LA-submodules of an LA-module is an LA-submodule.*

Theorem 4. *If A and B be LA-submodules of an LA-module M over an LA-ring with left identity 1, then $A + B$ is an LA-submodule of M.*

Corollary 4. *Let $(R, +, \cdot)$ be an LA-ring with left identity 1 and A, B are ideals of R, then $A + B$ is an ideal of R.*

Definition 3. Let M be an LA-module and $A \subset M$ be an LA-submodule. We define the quotient module or factor module M/A by $M/A = \{ A + m : m \in M\}$. That is, M/A is the set of equivalence classes of elements of M, where $m, n \in M$ are equivalent, if $m - n \in A$. An equivalence class is denoted by $A + m$ or by $[m]$. Each element in the class $A + m$ is called a representative of the class.

Lemma 2. *With the canonical operations, by choosing representatives, $(A + m) + (A + n) = A + (m + n)$, the set M/A is an LA-group. A, the equivalence class of $0 \in M$ is the left identity of M/A. The map $\pi : M \to M/A$, $\pi(m) = A + m$ is a surjective LA–group homomorphism.*

Proof. The proof is straight forward. We just show that the addition is well–defined (independent of the chosen representatives). For well-defined, let $A + m = A + m'$ and $A + n = A + n'$. Implies that $m \in A + m'$ and $n \in A + n'$. So $m = a + m'$ and $n = b + n'$ for some $a, b \in A$. Now, $m + n = (a + m') + (b + n') = (a + b) + (m' + n') \in A + (m' + n')$. So $A + (m + n) = A + (m' + n')$.

One can easily verify all the axioms of an LA-group. ∎

Proposition 2. *Let M be an LA-module over an LA-ring R and $A \subset M$ be an LA-submodule of M. Then, the set M/A is an LA-module.*

Proof. By lemma 2, M/A is an LA-group. Define a map $R \times M/A \longrightarrow M/A$ by $r(A+m) = A + rm$. For well-defined, let $A+m, A+m' \in M/A$ be such that $A+m = A+m'$, this means $m \in A+m'$ and therefore $m - m' \in A$. So, $r(m-m') \in A$ for all $r \in R$. That is, $rm - rm' \in A$, implies that $A + rm = A + rm'$.

One can easily verify all axioms of LA-module. Hence M/A is an LA-module over an LA-ring R. ∎

The LA-module of proposition 2 is called Quotient LA-module.

Corollary 5. *Let $(R, +, \cdot)$ be an LA-ring with left identity 1 and A be its left ideal. Then $R/I = \{I + r : r \in R\}$ is an LA-ring with left identity I.*

The LA-ring of corollary 5 is called Quotient LA-ring.

3. LA-Module Homomorphism

In this section, we define LA-module homomorphism and discuss fundamental theorems of homomorphisms for LA-modules.

Definition 4. (a) *L*et M,N be *LA*-modules over an LA-ring R. A map $\varphi : M \to N$ is called an LA-module homomorphism (or simply R-homomorphism) if, for all r in R and m, n in M (i) $\varphi(m+n) = \varphi(m) + \varphi(n)$, (ii) $\varphi(rm) = r\varphi(m)$.

(b) If $N = M$, then φ is called an endomorphism.

(c) If φ is one-one (resp. onto, bijective), then φ is called monomorphism (resp. epimorphism, isomorphism).

(d) M is said to be isomorphic to N, denoted by $M \cong N$, if there exists an isomorphism from M onto N.

Theorem 5. *Let $\varphi : M \longrightarrow N$ be an LA-module homomorphism from an LA-module M to an LA-module N, then*

(1) If A is an LA-submodule of M, then $\varphi(A)$ is an LA-submodule of N.

(2) If B is an LA-submodule of N, then $\varphi^{-1}(B)$ is an LA-submodule of M.

Corollary 6. *Let $f : R \longrightarrow S$ be an LA-ring epimorphism.*

(1) If I is an ideal of R, then $f(I)$ is an ideal of S.

Theorem 6. *(2) If J is an ideal of S, then $f^{-1}(J)$ is an ideal in R.*

Proposition 3. *If $\varphi : M \longrightarrow N$ is an LA-module homomorphism (particularly LA-ring homomorphism $f : R \longrightarrow S$), then*

(1) $\varphi(0_M) = 0_N$ ($f(0_R) = 0_S$).

(2) $\varphi(-m) = -\varphi(m)$ ($f(-r) = -f(r)$).

(3) $\varphi(m_1 - m_2) = \varphi(m_1) - \varphi(m_2)$ ($f(r_1 - r_2) = f(r_1) - f(r_2)$).

Definition 5. Let $\varphi : M \to N$ be an LA$-$module homomorphism. The kernel of φ, Ker(φ) is defined by $Ker\varphi = \{m \in M : \varphi(m) = 0\}$. The image of φ, Imφ, is defined by $Im\varphi = \{\varphi(m) | m \in M\}$.

Lemma 3. *$Ker\varphi$ and $Im\varphi$ are submodules of M and N, respectively.*

Proposition 4. *If $\varphi : M \longrightarrow N$ be an LA–module homomorphism from an LA-module M to an LA-module N, then φ is one-one if and only if $Ker\varphi = \{0\}$.*

Proposition 5. *Let M and N be LA-modules over an LA-ring R, the collection of all LA-module homomorphisms from M to N is denoted by $Hom_R(M,N)$. If we define addition and scaler multiplication as: $(\varphi_1 + \varphi_2)(m) = \varphi_1(m) + \varphi_2(m)$ and*
$(r\varphi)(m) = r\varphi(m)$, then $Hom_R(M,N)$ is an LA-module over R.

Remark 5. In module theory, $\mathrm{Hom}_R(M,M)$ is a ring with multiplication is composition of mappings. The ring is called endomorphism ring. In case of LA-modules, $\mathrm{Hom}_R(M,M)$ is not an LA-ring.

3.1. **Isomorphisms Theorems for LA-modules.**

Theorem 7. *Let $\theta : M \longrightarrow N$ be an LA-module homomorphism from an LA-module M to an LA-module N. If θ is an LA-module epimorphism, then $M/Ker\theta \cong N$.*

Corollary 7. *Let φ be an epimorphism of an LA-ring R to an LA-ring S. Then $R/\ker\varphi \cong S$.*

Remark 6. If H is an LA-subgroup of an LA-group G, then $H + a = H$ iff $a \in H$.

Theorem 8. *Let A and B be LA-submodules of an LA-module M over an LA-ring R. Then $(A + B)/A \cong B/(A \cap B)$.*

Proof. Since $A = A+0 \subseteq A+B$ and $A\cap B \subseteq B$ so $(A + B)/A$ and $B/(A \cap B)$ are well-defined. Now define a map $\varphi : (A + B) \longrightarrow B/C$, where $C = A \cap B$ by $\varphi(a + b) = C + b$. For well-defined, let $a_1 + b_1, a_2 + b_2 \in A + B$ be such that $a_1 + b_1 = a_2 + b_2$ implies that $(a_1 + b_1) - b_1 = (a_2 + b_2) - b_1$, so $(-b_1 + b_1) + a_1 = (-b_1 + b_2) + a_2$, by medial law. So

$$\begin{aligned} a_1 &= (-b_1 + b_2) + a_2, \text{ implies that} \\ a_1 - a_2 &= ((-b_1 + b_2) + a_2) - a_2 = (-a_2 + a_2) + (-b_1 + b_2), \text{ by medial law} \\ a_1 - a_2 &= -b_1 + b_2 \in A \cap B = C. \text{ That is } -b_1 + b_2 \in C \text{ implies that} \\ -(-b_1 + b_2) &= b_1 - b_2 \in C. \text{ This means } C + b_1 = C + b_2 \text{ and so} \\ \varphi(a_1 + b_1) &= \varphi(a_2 + b_2). \end{aligned}$$

It is easy to verify that φ is an epimorphism. By theorem 7, we have $(A + B)/Ker\varphi \cong B/C$.

Now we have to show that $Ker\varphi = A$. For this, let $a + b \in Ker\varphi$, then $\varphi(a + b) = C$, implies that $C + b = C$, so $b \in C = A \cap B$, implies that $b \in A$, hence $a + b \in A$, because A is an LA-module. This implies $Ker\varphi \subseteq A$.

Now let $a \in A$, then $\varphi(a) = \varphi((a + 0) + 0) = C + 0 = C$. This implies that $a \in Ker\varphi$, so $A \subseteq Ker\varphi$. Hence $(A + B)/Ker\varphi \cong B/C$, where $C = A \cap B$. ∎

Corollary 8. *If I and J are two ideals of an LA-ring R, then $(I+J)/I \cong J/(I \cap J)$.*

Theorem 9. *Let A and B be LA-submodules of an LA-module M over an LA-ring R. If $A \subseteq B$, then $M/B \cong (M/A)/(B/A)$.*

Corollary 9. *If R is an LA-ring and I, J are ideals in R, such that $I \subset J$, then $(R/I)/(J/I) \cong R/J$.*

We call theorems 7,8 and 9, the fundamental theorems of LA-module homomorphism and corollaries 7, 8, 9, the fundamental theorems of LA-ring homomorphism.

3.1.1. *Direct Sum.*

Definition 6. Let M be an LA-module over an LA-ring R. Let A and B be LA-submodules of M. Then M is said to be the internal direct sum of A and B, if every element $m \in M$ can be written in one and only one way as $m = a + b$, where $a \in A$ and $b \in B$. Symbolically the direct sum is represented by the notation $M = A \oplus B$.

Theorem 10. *Let M be an LA-module over an LA-ring R. If A and B are LA-submodules of M, then M is the internal direct sum of A and B if and only if*

(1) $M = A + B$.
(2) $A \cap B = \{0\}$.

Proof. For sufficient condition:

Suppose that (1) and (2) holds. Let $m \in M$ be such that $m = a_1 + b_1 = a_2 + b_2$, where $a_1, a_2 \in A$ and $b_1, b_2 \in B$, we have to show that $a_1 = a_2$ and $b_1 = b_2$. As $a_1 + b_1 = a_2 + b_2$, this implies that $a_1 = (a_2 + b_2) - b_1 = (-b_1 + b_2) + a_2$, so $a_1 - a_2 = -b_1 + b_2 \in A \cap B = \{0\}$. That is $a_1 - a_2 = 0$ and $-b_1 + b_2 = 0$. Hence $a_1 = a_2$ and $b_2 = b_1$.

For necessary condition:

Clearly, $A + B = M$. Now, let $x \in A \cap B$ then $x \in A$ and $x \in B$. Since $x = 0 + x \in A + B \subseteq M$ and $x = (x + 0) + 0 \in A + B \subseteq M$. Therefore, by uniqueness of x, $(x + 0) = 0$ and $x = 0$. This implies that $x = 0$. So $A \cap B = \{0\}$. ∎

Proposition 6. *Let $f : M \longrightarrow M$ be an LA-module homomorphism of an LA-module M, then $M = Kerf \oplus imgf$, if $f \circ f = f$.*

Proposition 7. *Let $f : M \longrightarrow N$ and $g : N \longrightarrow M$ be LA-module homomorphisms such that $g \circ f = I_M$, then $N = Kerg \oplus imgf$.*

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