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Numerical Solution of Integral Equations with Legendre Basis

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Abstract

In this article first we considered a class of Fredholm integral equation of the first kind which arise in a large number of problems in applied mathematics.Then we used Legendre functions as base for projection method to estimate the solution of the integral equation. Finally, by using numerical examples compared our estimation with Wavelet-Legendre as a basis. Algorithms was coded in Maple 13 and run on a personal Pentium IV system, 2.00 GHz, 2048MB RAM.

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1 Introduction

In this paper, we solve Fredholm integral equations of the first kind given in the form

$$\int_0^1 k(t,s)f(s)ds = g(t) \qquad t \in [0,1] \tag{1}$$

One major difficulty in working with any Fredholm integral equation of the first kind is that the solution does not depend continuously on the given function $g(t)$. Furthermore this instability carries over to the solution of the algebraic system arising from discretization of the integral equation[1]. So for simplicity and better understanding of the problem (1) we suppose, the function $g \in L^2[0,1]$, the kernel $k \in L^2([0,1] \times [0,1])$ are given and $f \in L^2[0,1]$ is the unknown function to be determined. Here we would like to use orthogonal Legendre functions as a basis in Galerkin's methods for numerical solution of Fredholm integral equations of the first kind.

2 Projection method

For numerical solving of integral equation (1) we should choose a family of functions with finite dimension. In practice, we choose a sequence of finite dimensional subspaces $X_n \subset X, n \geq 1$, with X_n having dimension d_n. Let X_n have a basis $\{\varphi_1, \varphi_2, ..., \varphi_d\}$,with $d \equiv d_n$ for notational simplicity. We seek a function $f_n \subset X_n$, and it can be written as

$$f_n(s) = \sum_{j=1}^{d} c_j \varphi_j(s) \qquad s \in [0, 1]$$

This is substituted into (1), and the coefficients $\{c_1, c_2, ..., c_d\}$ are determined by forcing the equation to be almost exact in some sense. For later use, introduce $r_n(t) = \int_0^1 k(t, s) f_n(s) ds - g(t) = \sum_{j=1}^{d} c_j \{\int_0^1 k(t, s)\varphi_j(s)ds\} - g(t) \qquad t \in [0, 1]$ This is called the residual in the approximation of the equation when using $f \approx f_n$. The coefficients $\{c_1, c_2, ..., c_d\}$ are chosen by forcing $r_n(t)$ to be approximately zero in some sense. The hope are that the resulting $f_n(s)$ function will be a good approximation of the true solution $f(s)$. The elements $\{\varphi_1, \varphi_2, ..., \varphi_d\}$ are the orthogonal basis functions defined on a certain interval $[0, 1]$. Here we choose $\varphi_i(t)$, as Legendre on $[0, 1]$.

2.1 Galerkin methods

We identify inner product $< x, y >= \int_0^1 x(t)y(t)dt$. Now to determine unknown coefficients we impose the following requirements:

$$< r_n, \varphi_i >= 0$$

This yields the linear system

$$\sum_{j=1}^{d} c_j \int_0^1 \{\int_0^1 k(t, s)\varphi_j(s)ds\}\varphi_i(t)dt = \int_0^1 g(t)\varphi_i(t)dt \qquad i = 1, 2, ..., d_n$$

or $AC = B$, where $A = [A_{ij}]_{n\times n}, C = [c_1, c_2, ..., c_d]^t$ and $B = [\beta_1, \beta_2, ..., \beta_d]^t$

$$A_{ij} = \int_0^1 \int_0^1 k(t, s)\varphi_j(s)\varphi_i(t)dsdt$$

and

$$\beta_i = \int_0^1 g(t)\varphi_i(t)dt.$$

2.2 Collocation methods

Pick distinct node points $t_1, t_2, ..., t_d \in [0, 1]$, and require

$$r_n(t_i) = 0, \quad for \quad i = 1, 2, ..., d_n$$

This leads to determining $\{c_1, c_2, ..., c_d\}$ as the solution of the linear system

$$\sum_{i=1}^{d} c_j\{\int_0^1 k(t_i, s)\varphi_j(s)ds\} = g(t_i) \qquad i = 1, 2, ..., d_n$$

In the operator form this yields linear system

$$\sum_{i=1}^{d} c_j(\kappa\varphi_j)(t_i)ds = g(t_i) \qquad i = 1, 2, ..., d_n$$

where

$$(\kappa\varphi_j)(t_i) = \int_0^1 k(t_i, s)\varphi_j(s)ds.$$

In this paper colloction points are $t_i = \frac{i}{n}$, for $i = 1, ..., d_n$ so that we have a system of linear equations $A_nX = b_n$ where

$$A_n = [\int_0^1 K(t_i, s)\varphi_j(s)ds]_{i=0}^{n}, \quad j = 1, ..., d_n$$

$$b_n = [g(t_j)], \quad j = 1, ..., d_n$$

3 Legendre polynomial

Consider the well-known Legendre polynomials of order N, $L_N(s)$, which are orthogonal on the interval $[-1, 1]$ with respect to the weight function $w(s) = 1$ and derived from the following recursive formula:

$$L_{N+1}(s) = \frac{2N + 1}{N + 1} sL_N(s) - \frac{N}{N + 1} L_{N-1}(s); N = 1, 2, 3..., d$$

In this case the initial condition is $L_0 = 1, L_1 = s$.[2] In order to use the legendre basis we transfer the s-interval $[-1, 1]$ into t-inteval $[0, 1]$ by means of transformations $t = \frac{s}{2} + \frac{1}{2}$.

4 Numerical examples

In this paper for showing the efficiency of the Galerkin method for solving Fredholm Integral Equations, here we present our numerical experiments by some example.

Numerical results of Example 1

t	$exact$	LGM	WGM
0	1	0.99849	0.99994
0.1	1.10517	1.10504	1.10519
0.2	1.22140	1.22165	1.22148
0.3	1.34986	1.34999	1.34912
0.4	1.49182	1.49173	1.49185
0.5	1.64872	1.64851	1.64795
0.6	1.82212	1.82201	1.82296
0.7	2.01375	2.01390	2.01223
0.8	2.22554	2.22580	2.22467
0.9	2.45960	2.45946	2.44356

Example 4.1

$$\int_0^1 k(t,s)f(s)ds = e^t + (1-e)t - 1$$

with exact solution $f(t) = e^t$ *and*

$$k(t,s) = \begin{cases} s(t-1), & t > s \\ t(s-1), & t \leq s \end{cases}$$

Example 4.2

$$\int_0^1 (t^2 + s^2)^{\frac{1}{2}} f(s)ds = \frac{(1+t^2)^{\frac{3}{2}} - t^3}{3}$$

with exact solution $f(t) = t$.

Numerical results of Example 2

t	$exact$	LGM	WGM
0	0.0000	0.0000	0.0000
0.1	0.1000	0.1000	0.1000
0.2	0.2000	0.2000	0.2000
0.3	0.3000	0.3000	0.3000
0.4	0.4000	0.4000	0.4000
0.5	0.5000	0.5000	0.5000
0.6	0.6000	0.6000	0.6000
0.7	0.7000	0.7000	0.7000
0.8	0.8000	0.8000	0.8000
0.9	0.9000	0.9000	0.9000

Example 4.3

$$\int_0^1 e^{(t+1)s} f(s)ds = \frac{1 - e^{t+1}}{(t+1)^2} + \frac{e^{(t+1)}}{t+1}$$

with exact solution $f(t) = t$.

Table 1 and 2,3 contain a numerical comparison between our solution using Legendre-Galerkin method and the solution of the same problems presented in [3,4].

Numerical results of Example 3

t	*exact*	*LGM*	*WGM*
0.1	0.1000	0.1000	0.10295
0.2	0.2000	0.2000	0.19398
0.3	0.3000	0.3000	0.30222
0.4	0.4000	0.4000	0.39634
0.5	0.5000	0.5000	0.49128
0.6	0.6000	0.6000	0.60241
0.7	0.7000	0.7000	0.69732
0.8	0.8000	0.8000	0.80670
0.9	0.9000	0.9000	0.89840

5 Conclusion

In this work, we proposed the LGM for solving the linear integral equations and compared our results with the presented results in [3,4]. Moreover, it minimizes the computational calculus and supplies quantitatively reliable results. But WGM has a complicated computational calculus and is not easy.[3,4]

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