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Stability Conditions of L_4 in the RTBP when the Smaller Primary and the Infinitesimal Body Are Oblate Spheroids

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Abstract

The non linear stability conditions for the triangular point L_4 in the restricted three body problem are found in terms of the parameters of motion: the mass ratio μ of the primary bodies and the moments of inertia of the smaller primary and the infinitesimal mass, which are both taken as oblate spheroids.

Keywords: Restricted Three Body Problem – L_4 Stability

1. Introduction

The restricted problem of 3 bodies possesses 5 equilibrium points, 3 collinear L_1, L_2 and L_3 and two triangular points L_4 and L_5. In the linear sense the collinear points are unstable for any value of the mass ratios, while L_4 and L_5 are stable if the mass ratio μ of the finite bodies is less than μ_0=0.03852… (Szebehely, 1967a).

Duboshin (1960) found particular solutions for the problem of translational-rotational motion where the shapes were taken into account. Danby (1965) considered Lagrange's solution of the three body problem taking one of the bodies as an oblate spheroid. Kondurar and Shinkarik (1971) extended Duboshin's work for the circular restricted problem of three bodies. Subba Rao and Sharma (1975) considered the problem with one of the primaries as an oblate spheroid and the equatorial plane coinciding with the plane of motion. Choudhary (1977) extended these results for the generalized elliptic restricted problem of three bodies. Ishwar (1997) studied the stability of L_4 where both the infinitesimal mass and one of the primaries have been taken as oblate spheroids. Khanna and Bhatnagar(1998) studied the linear stability of L_4 in the restricted 3-body problem when the smaller primary is a triaxial rigid body. Hallane et. Al. (2000) studied the non linear stability of L_4 when the bigger primary is a triaxial and its equatorial plane is coinciding.

In this paper, we follow Ishwar (1997) in studying L_4 stability when both the infinitesimal mass and one of the primaries have been taken as oblate spheroids, but the second order stability conditions here are found by a simpler way.

The present analysis is important for many astrophysical problems, in fact it holds for any binary system with the secondary accompanied with a third smaller body (eg. a star or a planet). In particular, the procedure is applicable to the sun-earth-moon system directly, since both the earth and the moon are oblate. But in this case the fact that the potential of the moon is not earth-like is to be accounted for(in this case J_2, J_{22}, J_3, J_4 and J_5 are all in the same order 10^{-4}). Also many (if not all) triple close stellar systems satisfy the conditions to apply the present procedure. This is in particular usual in triple pulsars (Trenti et. al., 2005). Clearly the case when all or two of the triple are spherical is a special case in which the inertia moments are taken to be equal.

2. The Lagrangian

The Lagrangian of the problem is given by,

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - (\dot{x}y - x\dot{y}) + \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{C-A}{2}\left(\frac{1-\mu}{r_1{}^3} + \frac{\mu}{r_2{}^3}\right) + \frac{C_2 - A_2}{2r_2{}^3} \tag{1}$$

where,

$$r_1^{\;2} = (x+\mu)^2 + y^2$$
$$r_2^{\;2} = (x-1+\mu)^2 + y^2 \qquad (2)$$

A and *C* are the small and large moments of inertia respectively of the infinitesimal body of mass *m* divided by *m*, where A_2, and C_2 are the normalized equatorial and polar moments of inertia respectively of the smaller primary of mass m_2 divided by m_2. The mean motion *n* is taken to be unity for the sake of simplicity.

The coordinates of the triangular equilibrium points L_4 and L_5 are:

$$x = \frac{1}{2} - \mu, \qquad y = \pm\sqrt{r^2 - \frac{1}{4}} \qquad (3)$$

The value of *r* is given by Kondurar and Shinkarik (1971) through the relation,

$$r^5 - r^2 - \frac{3}{2}(C-A) - \frac{3}{2}(C_2 - A_2)\mu = 0 \qquad (4)$$

This equation is the condition for the existence of isosceles triangular equilibrium points different from the classical case.

3. Second Order Stability of Motion Around L_4

In order to study the non-linear stability around the point L_4, the origin is shifted to it. Then we expand the Lagrangian in the coordinates *x* and *y*. We get the following orders of the Lagrangian function,

$$L_0 = \frac{(1-2\mu)^2}{8} + \frac{4r^2-1}{8} + \frac{1}{r} + \frac{C-A}{2r^3} + \frac{(C_2-A_2)\mu}{2r^3} \qquad (5)$$

$$L_1 = x\left(\frac{1-2\mu}{2} - \frac{1}{2r^3} + \frac{\mu}{r^3} - \frac{3(C-A)}{4r^5} + \frac{3(C-A)\mu}{2r^5} + \frac{3(C_2-A_2)\mu}{4r^5}\right)$$
$$+ \sqrt{4r^2-1}\, y\left(\frac{1}{2} - \frac{1}{2r^3} - \frac{3(C-A)}{4r^5} - \frac{3(C_2-A_2)\mu}{4r^5}\right) + \frac{1-2\mu}{2}\dot{y} - \frac{\sqrt{4r^2-1}}{2}\dot{x} \qquad (6)$$

The second order part of the Lagrangian is found to be

$$L_2 = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - (\dot{x}y - \dot{y}x)$$
$$+ x^2(\frac{1}{2} - \frac{1}{2r^3} + \frac{3}{8r^5} - \frac{3(C-A)}{4r^5} + \frac{15(C-A)}{16r^7} - \frac{3(C_2 - A_2)\mu}{4r^5} + \frac{15(C_2 - A_2)\mu}{16r^7})$$
$$+ y^2(\frac{1}{2} - \frac{1}{2r^3} + \frac{3(4r^2-1)}{8r^5} - \frac{3(C-A)}{4r^5} + \frac{15(C-A)(4r^2-1)}{16r^7} - \frac{3(C_2 - A_2)\mu}{4r^5}$$
$$+ \frac{15(C_2 - A_2)(4r^2-1)\mu}{16r^7}) + \sqrt{4r^2-1}\; xy(\frac{3}{4r^5} - \frac{3\mu}{2r^5} - \frac{15(C_2 - A_2)\mu}{8r^7} + \frac{15(C-A)(1-2\mu)}{8r^7}) \quad (7)$$

The last term is different with Ishwar (1997)

This can be rearranged as

$$L_2 = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - (\dot{x}y - \dot{y}x) + \alpha\, x^2 + \beta\, y^2 + \gamma\, xy \quad (8)$$

Lagrange's equations of the second order will be

$$\frac{d}{dt}\frac{\partial L_2}{\partial \dot{x}} - \frac{\partial L_2}{\partial x} = 0 \qquad \frac{d}{dt}\frac{\partial L_2}{\partial \dot{y}} - \frac{\partial L_2}{\partial y} = 0$$

This will give

$$\ddot{x} = 2\dot{y} + 2\alpha\, x + \beta\, y$$
$$\ddot{y} = -\,2\dot{x} + \beta\, x + 2\gamma\, y \quad (9)$$

4. Stability Conditions in terms of k_1, k_2 and μ

The conditions of stability are found from the characteristic equation

$$\lambda^4 + (4 - 2\gamma - 2\alpha)\lambda^2 + 4\alpha\gamma - \beta^2 = 0$$

Since λ has to be pure imaginary, then λ^2 should be a real negative which will be satisfied only when,

$$(4 - 2\gamma - 2\alpha) > 0, \qquad 4\alpha\gamma - \beta^2 > 0, \qquad (4 - 2\gamma - 2\alpha)^2 > 4(4\alpha\gamma - \beta^2)$$

In order to get the above conditions in terms of the parameters of the motion (μ and the moments of inertia of the masses *m* and m_2) we have to find a solution of eq(4) to get *r* in terms of the mentioned parameters. We expand *r* as a power series in the small parameter μ ,

$$r = 1 + a_1\mu + a_2\mu^2 + a_3\mu^3 + \dots \tag{10}$$

We introduce the constants k_1 and k_2 defined by

$$C - A = k_1\mu, \qquad C_2 - A_2 = k_2\mu \tag{11}$$

This will facilitate the expansion of *r* in terms of the small quantity μ , where k_1 and k_2 are constants of zero order characterizing the system with μ .

α, β and γ are now

$$\alpha = \frac{1}{2} - \frac{1}{3r^3} + \frac{1}{r^5}\left(\frac{3}{8} - \frac{3k_1\mu}{4} - \frac{3k_2\mu^2}{4}\right) + \frac{15}{16r^7}(k_1\mu + k_2\mu^2) \tag{12.1}$$

$$\beta = \frac{1}{2} + \frac{1}{3r^3} + \frac{1}{r^5}\left(\frac{-3}{8} + 3k_1\mu + 3k_2\mu^2\right) - \frac{15}{16r^7}(k_1\mu + k_2\mu^2) \tag{12.2}$$

$$\gamma = \frac{\sqrt{4r^2 - 1}}{4}\left\{\frac{1}{r^5}(3 - 6\mu) + \frac{15}{2r^7}(k_1\mu - 2k_1\mu^2 - k_2\mu^2)\right\} \tag{12.3}$$

Eq (4) is now,

$$r^5 - r^2 - \frac{3}{2}k_1\mu - \frac{3}{2}k_2\mu^2 = 0 \tag{13}$$

Substituting of *r* in (13) and equating the coefficients of similar powers of μ , we get

$$a_1 = \frac{1}{2}k_1 \tag{14.1}$$

$$a_2 = \frac{1}{2}k_2 - \frac{3}{4}k_1^2 \tag{14.2}$$

$$a_3 = \frac{11}{6}k_1^3 - \frac{3}{2}k_1k_2 \tag{14.2}$$

Thus,

$$r = 1 + \frac{k_1}{2}\mu + (\frac{k_2}{2} - \frac{3k_1^2}{4})\mu^2 + O(\mu^3) \tag{15}$$

Using the above expression to expand the involving powers of *r*, we get the required expansions,

$$\sqrt{4r^2 - 1} = \sqrt{3}(1 + \frac{2\mu k_1}{3} + \mu^2(-\frac{19k_1^2}{18} + \frac{2k_2}{3})) \tag{16.1}$$

$$r^{-2} = 1 - \mu k_1 + \mu^2(\frac{9k_1^2}{4} - k_2) \tag{16.2}$$

$$r^{-3} = 1 - \frac{3\mu k_1}{2} + \frac{1}{4}\mu^2(15k_1^2 - 6k_2) \tag{16.3}$$

$$r^{-5} = 1 - \frac{5\mu k_1}{2} + \frac{1}{2}\mu^2(15k_1^2 - 5k_2) \tag{16.4}$$

$$r^{-7} = 1 - \frac{7\mu k_1}{2} + \frac{1}{4}\mu^2(49k_1^2 - 14k_2) \tag{16.5}$$

Applying the expansions of the powers of *r*, we arrive at,

$$\alpha = \frac{3}{8} - \frac{15}{32}k_1^2\mu^2 \tag{17.1}$$

$$\beta = \frac{9}{8} + \frac{39}{16}k_1\mu - (\frac{15}{16}k_1 + \frac{105}{16}k_1^{\,2} - \frac{3}{2}k_2)\mu^2 \tag{17.2}$$

$$\gamma = \sqrt{3}\{\frac{3}{4} + (\frac{1}{2}k_1 - \frac{3}{2})\mu - (k_1 + \frac{83}{48}k_1^2 - \frac{13}{4}k_2)\mu^2\} \tag{17.3}$$

Substituting Eqs (17) in the stability conditions, we finally get them as,

$$\frac{13}{8} - \frac{3\sqrt{3}}{4} + \frac{3\sqrt{3}\mu}{2} + (\frac{-\sqrt{3}\mu}{2} + \sqrt{3}\mu^2)k_1 + (\frac{15\mu^2}{32} + \frac{83\mu^2}{16\sqrt{3}})k_1^2 + \frac{11}{4}\sqrt{3}\mu^2 k_2 > 0 \tag{18.1}$$

$$\frac{-81}{64} + \frac{9\sqrt{3}}{8} - \frac{9\sqrt{3}\mu}{4} + (\frac{3\sqrt{3}\mu}{4} - \frac{351\mu}{64} + \frac{135\mu^2}{64} - \frac{3\sqrt{3}\mu^2}{2})k_1 + (\frac{2259}{256} - 4\sqrt{3})\mu^2 k_1^2 - (\frac{27}{8} - \frac{39\sqrt{3}}{8})\mu^2 k_2 > 0 \tag{18.2}$$

$$\frac{179}{8}-\frac{57\sqrt{3}}{4}-27\mu+\frac{57\sqrt{3}}{2}\mu+27\mu^2+(\frac{495\mu}{16}-\frac{19\sqrt{3}\mu}{2}-\frac{711\mu^2}{16}+19\sqrt{3}\mu^2)k_1 + (\frac{107\sqrt{3}}{3}-\frac{3669}{64})\mu^2 k_1^2+(\frac{247\sqrt{3}}{4}-45)\mu^2 k_2>0 \tag{18.3}$$

5. Motion About L_4

When the conditions of stability are satisfied, the eigen values can be written as,

$$\lambda_{1,2}=\pm i\theta_1, \qquad \lambda_{3,4}=\pm i\theta_2$$

where,

$$\theta_1=\frac{1}{\sqrt{2}}\sqrt{\beta^2+\sqrt{\beta^2-4\gamma}} \tag{19.1}$$

$$\theta_2=\frac{1}{\sqrt{2}}\sqrt{\beta^2-\sqrt{\beta^2-4\gamma}} \tag{19.1}$$

which can be expressed in the parameters of the motion as,

$$\theta_1=3\mu k_1\frac{18363+17728\sqrt{3}+(81789-44096\sqrt{3})\mu}{16384\sqrt{2}}-\frac{153855+57472\sqrt{3}}{65536\sqrt{2}}\mu^2 k_1^2 + \frac{3(-10763-14208\sqrt{3}+768(192+37\sqrt{3})\mu-147456\mu^2+16(23271+5576\sqrt{3})\mu^2 k_2}{32768\sqrt{2}} \tag{20.1}$$

$$\theta_2=3\mu k_1\frac{41541-17728\sqrt{3}+(44096\sqrt{3}-104829)\mu}{16384\sqrt{2}}+\frac{57472\sqrt{3}-1002753}{65536\sqrt{2}}\mu^2 k_1^2 + \frac{3(38411+14208\sqrt{3}-768(192+37\sqrt{3})\mu+147456\mu^2-16(18663+5576\sqrt{3})\mu^2 k_2}{32768\sqrt{2}} \tag{20.2}$$

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