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Maximal Independent Sets without Including any Leaves in Trees[1]

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Abstract

In this paper we study maximal independent sets in trees without including any leaves. In particular, we determine some small and the largest number of these sets in trees. Extremal trees achieving these values are determined too.

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1 Introduction

Let G be a graph. A subset S of $V(G)$ is *independent* if no two vertices in S are adjacent. A *maximal independent set* is an independent set that is not a proper subset of any other independent set. For convenience, we write a mi-set for a maximal independent set of G. The number of maximal independent sets of G is denoted by $mi(G)$. Denote by $sub_{xy}(G)$ the graph obtained from G by subdividing the edge xy of G.

One reason why upper bounds on $\mathrm{mi}(G)$ are of interest is that better estimates on the size of mi-sets lead to improvements on the time analysis of algorithms determining several hard graph invariants. The literature includes many papers dealing with counting independent sets in graphs. Erdős and Moser raised the problem of determining the maximum value of $\mathrm{mi}(G)$ for

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a general graph of order n and the extremal graphs achieving the maximum value. This problem was solved by Moon and Moser [20]. Since then, researchers have studied the problem for many special graph classes, see $[1-7, 9, 10, 12-16, 18-26]$. For other related, including algorithmic, results on $\mathrm{mi}(G)$, see [11, 8, 13, 17].

Along all the results, an interesting problem is to consider the bounds of the number of mi-sets in trees. As pointed in [10], though it remains open, it would be interesting to see how assumptions on some further basic properties of graphs, for example, girth, maximum degree, etc. Wloch [24] gave some solutions for mi-sets including the set of leaves in trees. In this paper, from the idea of [24], we consider the extremal number of mi-sets in trees without including any leaves.

2 Preliminaries

Let T be a tree. For $x \in V(T)$, denote by $L(x)$ the set of leaves attached to the vertex x. A vertex $x \in V(T)$ with $L(x) \neq \emptyset$ is called a *support vertex*. If $|L(x)| \geq 2$ then x is called a *strong support vertex*. If $|L(x)| = 1$ then x is called a *weak support vertex*, and the unique leaf attached to x is called a *single leaf*. Throughout the paper, denote by L the set of leaves and by M the set of support vertices of T. Denote by $\mathrm{mi}_{-L}(G)$ the number of mi-sets including the set of support vertices. Clearly, $\mathrm{mi}_{-L}(G)$ is just the number of mi-sets without including any leaves.

The following lemmas are clear, and we omit the details.

Lemma 2.1 *Let T be a tree and S be a mi-set of T. If $x \in M$ and $S \cap L(x) \neq \emptyset$, then $L(x) \subseteq S$.*

Lemma 2.2 *Let T be a tree and x be a strong support vertex of T. For any $L'(x) \subset L(x)$, $mi_{-L}(T) = mi_{-L}(T \backslash L'(x))$.*

Lemma 2.3 *Let T be a tree and S be a mi-set of T. If $S \cap L(T) = \emptyset$, then $M \subseteq S$.*

Lemma 2.4 *Let T be a tree. If T has a mi-set without including any leaves, then any two support vertices are not adjacent.*

Lemma 2.5 *Let T be a tree. If T has a mi-set without including any leaves, and $mi_{-L}(T) = p$ (p is prime), then $T \backslash N[M]$ has exactly one nontrivial component.*

3 Trees with the smallest numbers of mi-sets without including any leaves

Theorem 3.1 *Let T be a tree with order $n \geq 3$. Then $mi_{-L}(T) = 1$ if and only if $T \backslash N[M]$ is an empty graph or $V(T) = N[M]$.*

Theorem 3.2 *Let T be a tree with order $n \geq 3$. Then $mi_{-L}(T) = 2$ if and only if $T \backslash N[M]$ contains only one nontrivial component $K_{1,t}$, $t \geq 1$.*

Proof: For any mi-set S without including any leaves in T, from Lemma 2.3, we have $M \subseteq S$. Note that, for any $S \subset V(T)$ with $M \subseteq S$, S is a mi-set without including any leaves of T if and only if $S \backslash M$ is a mi-set of $T \backslash N[M]$.

If $T \backslash N[M]$ contains only one nontrivial component $K_{1,t}$, $t \geq 1$, clearly, we have $\mathrm{mi}_{-L}(T) = \mathrm{mi}(T \backslash N[M]) = 2$.

Conversely, assume that $\mathrm{mi}_{-L}(T) = 2$. Then $\mathrm{mi}(T \backslash N[M]) = 2$. Let $\tilde{T}$ be a subgraph of $T^{'}$ such that $T^{'} \backslash V(\tilde{T})$ consists of isolated vertices. Then $\mathrm{mi}(\tilde{T}) = \mathrm{mi}(T^{'}) = \mathrm{mi}_{-L}(T) = 2$. From Lemma 2.5, $\tilde{T}$ is a tree. We shall prove that $\tilde{T} = K_{1,t}$, $t \geq 1$.

Suppose that $\tilde{T} \ncong K_{1,t}$, $t \geq 1$. Then $\tilde{T}$ contains a path P_4. From $\mathrm{mi}(P_4) = 3$, we have $\mathrm{mi}(\tilde{T}) \geq 3$. A contradiction.

This completes the proof.

Let $\mathrm{I}_r = \{T :\ T \text{ is a tree and } \mathrm{mi}(T) = r\}$. Define a *m-star* $T^m_{i,j}$, $i, j \geq 1$, obtained from the path P_m, $m \geq 2$, by attaching i, j, respectively, leaves to the endpoints. When $m = 2$, we call it a *double star*. Denote by $B(3; i, j, s)$, $i, j, s \geq 1$ the tree obtained from a path P_3 by attaching i, j, s leaves to each vertex in P_3. Denote by $Q(4; i, j, s)$, $i, j, s \geq 1$ the tree obtained from a path P_4 by attaching i, j leaves to the endpoints of P_4, and attaching s leaves to an arbitrary interior vertex in P_4. In general, we have the following result. The proof is almost similar to the previous theorem.

Theorem 3.3 *Let T be a tree with order $n \geq 3$. Then $mi_{-L}(T) = p$ (p is a prime) if and only if $T \backslash N[M]$ contains only one nontrivial component in I_p.*

When p is small prime, we can list all the possible cases as following: $\mathrm{I}_2 = \{K_{1,t}\}$, $\mathrm{I}_3 = \{T^2_{i,j}\}$, $\mathrm{I}_5 = \{T^4_{i,j}, B(3; i, j, s)\}$, $\mathrm{I}_7 = \{T^5_{i,j}\}$. But for composite number r, the thing becomes complicated. For small composite number, we have the following result.

Theorem 3.4 *Let T be a tree with order $n \geq 3$. Then $mi_{-L}(T) = 4$ if and only if $T \backslash N[M]$ contains only either one nontrivial component $T^3_{i,j}$ or two nontrivial star components.*

Theorem 3.5 *Let T be a tree with order $n \geq 3$. Then $mi_{-L}(T) = 6$ if and only if $T \backslash N[M]$ contains only either one nontrivial component $Q(4; i, j, s)$ or two nontrivial components $K_{1,t}$ and $T^2_{i,j}$.*

4 Trees with the largest number of mi-sets without including any leaves

Firstly we give theorems which will provide a way to transform gradually a tree into a path of same order, such that the number of mi-sets without including any leaves does not decrease at each step.

A vertex x of tree T is *penultimate* if x is adjacent to at least $d(x) - 1$ leaves. Note that x is adjacent to $d(x)$ leaves if and only if T is a star and x is the center.

Lemma 4.1 *Let x be a weak support vertex of T with $L(x) = z$. If x is not a penultimate vertex, then $mi_{-L}(T) \leq mi_{-L}(T \backslash \{z\})$.*

Proof: Denote $T' = T \backslash \{z\}$. Let F_{-L} and F'_{-L} be families of all mi-sets without including any leaves in T and T', respectively. Clearly, we have that $\mathrm{mi}_{-L}(T) = |\mathrm{F}_{-L}|$ and $\mathrm{mi}_{-L}(T') = |\mathrm{F}'_{-L}|$. For any $S \in \mathrm{F}_{-L}$, S is also a mi-set without including any leaves in T', i.e., $\mathrm{F}_{-L} \subseteq \mathrm{F}'_{-L}$.

Lemma 4.2 *Let T be a tree with order $n \geq 3$. Let x be a strong support vertex with $L(x) = z_1, z_2, ...z_p$, $p \geq 2$. Let u be a penultimate vertex of T and $v \in N(u) \backslash L(u)$. Then $mi_{-L}(T) \leq mi_{-L}(sub_{[u,v]}(T \backslash \{z_i\}))$, $1 \leq i \leq p$.*

Proof: From Lemma 2.1, we have that $\mathrm{mi}_{-L}(T) = \mathrm{mi}_{-L}(T \backslash \{z_i\})$. Let $T' = sub_{[u,v]}(T \backslash \{z_i\})$. Let F_{-L} and F'_{-L} be families of all mi-sets without including any leaves in T and T', respectively.

Clearly, we have that $\mathrm{mi}_{-L}(T) = |\mathrm{F}_{-L}|$ and $\mathrm{mi}_{-L}(T') = |\mathrm{F}'_{-L}|$. For any $S \in \mathrm{F}_{-L}$, $v \notin S$. Moreover, we have that exactly one of S and $S \cup \{v\}$ must be a mi-set without including any leaves in T'. Clearly, we have the desired result.

Lemma 4.3 *Let T be a tree with order $n \geq 5$. Suppose that T contains two subtrees P_t and P_m, $t, m \geq 3$, such that*
(i) *P_t and P_m has an endpoint vertex x in common;*
(ii)*) the other endpoints of P_t and P_m are leaves in T;*
(iii) *each the internal vertices of P_t and P_m has degree two in T.*
Then $mi_{-L}(T) \leq mi_{-L}(T - xv + uv)$, where u is the endpoint vertex of P_m, and $v = N(x) \cap V(P_t)$.

Proof: Denote $T^{'} = T - xv + uv$. Let F_{-L} and $\mathrm{F}^{'}_{-L}$ be families of all mi-sets without including any leaves in T and $T^{'}$, respectively. Let $S \in \mathrm{F}_{-L}$ and $N(u) = w$. Then $u \notin S$ and $w \in S$. Clearly, S is independent in $T^{'}$. Next, we extend S to a mi-set without including any leaves in $T^{'}$ according to the following rules.

R1. $v \in S$. Then $x \notin S$.
(1.1) If $N(x)\backslash\{v\} \cap S = \emptyset$, then $S \cup \{x\} \in \mathrm{F}^{'}_{-L}$.
(1.2) If $N(x)\backslash\{v\} \cap S \neq \emptyset$, then $S \in \mathrm{F}^{'}_{-L}$.
R2. $v \notin S$. Let $a = N(v)\backslash\{x\}$.
(2.1) If $a \in S$, then $S \in \mathrm{F}^{'}_{-L}$.
(2.2) If $a \notin S$, then $S \cup \{v\} \in \mathrm{F}^{'}_{-L}$.

According to the rules above, any two distinct $S_1, S_2 \in \mathrm{F}_{-L}$ will be extended to different mi-sets without including any leaves in $T^{'}$. Thus we have that $\mathrm{mi}_{-L}(T) \leq \mathrm{mi}_{-L}(T - xv + uv)$. This completes the proof.

From Lemmas 4.1, 4.2 and 4.3, we have the following result.

Theorem 4.4 *Let T be a tree with order $n \geq 3$, then $mi_{-L}(T) \leq mi_{-L}(P_n)$.*

Finally, we present the recursive formula for the number of mi-sets without including any leaves in paths.

Theorem 4.5 *For any $n \geq 4$, $mi_{-L}(P_n) = mi_{-L}(P_{n-2}) + mi_{-L}(P_{n-3})$.*

5 Concluding remarks

Finally, we present the local augmentations in [24] which also preserves the number $\mathrm{mi}_{-L}(T)$ in tree T.

Let $\tilde{T}$ be an arbitrary tree, T-*addition* stands for a local augmentation of T which is the operation $T \rightarrow ad_{\tilde{T}(x,y)}(T)$ of adding to the vertex $x \in V(T)$ a graph $\tilde{T}$ so that a vertex x is identified with a fixed vertex $y \in V(\tilde{T})$. If $\tilde{T}$ is a star then we call this operation a *star-addition*.

Theorem 5.1 *Let T be a tree with order $n \geq 3$, and $\tilde{T}$ be a tree with a unique mi-set without including any leaves. Then $mi_{-L}(ad_{\tilde{T}(x,y)}(T)) = mi_{-L}(T)$ where x is an arbitrary support vertex in T and y is an arbitrary support vertex in $\tilde{T}$.*

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