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Growth Properties of Composite Entire Functions of Two Complex Variables

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Abstract

In the paper we study the comparative growth properties of composite entire functions of two complex variables. We also introduce the notion of zero order of entire functions of two complex variables and discuss some related results.

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1 Introduction, Definitions and Notations.

Let $f(z_1,z_2)$ be a non-constant entire function of two complex variables z_1 and z_2, holomorphic in the closed polydisc

$$\{(z_1,z_2):|z_j|\leq r_j,\ j=1,2 \text{ for all } r_1\geq 0, r_2\geq 0\ \}$$

$$\text{Let } F(r_1,r_2)=\max\{\,|f(z_1,z_2)|:|z_j|\leq r_j,\ j=1,2\}.$$

Then by the Hartogs theorem and maximum principle $\{[2],p.21,p.51\}$, $F(r_1,r_2)$ is an increasing function of r_1,r_2. The order $\rho=\rho(f)$ of $f(z_1,z_2)$ is defined {[2], p.338} as the infimum of all positive numbers μ for which

$$F(r_1,r_2)<\exp[(r_1r_2)^{\mu}]$$

holds for all sufficiently large value of r_1 and r_2.In other words

$$\rho(f)=\inf\{\mu>0:F(r_1,r_2)<\exp[(r_1r_2)^{\mu}] \text{ for all } r_1\geq R(\mu),r_2\geq R(\mu)\}.$$

Equivalent formula for $\rho(f)$ is {[2], p.339; [1]}

$$\rho(f)=\limsup_{r_1,r_2\to\infty}\frac{\log^{[2]}F(r_1,r_2)}{\log(r_1r_2)}$$

$$\text{where } \log^{[k]}x=\log(\log^{[k-1]}x) \text{ for } k=1,2,3,...\text{and } \log^{[0]}x=x.$$

Similarly the lower order $\lambda=\lambda(f)$ of $f(z_1,z_2)$ is defined as

$$\lambda(f)=\liminf_{r_1,r_2\to\infty}\frac{\log^{[2]}F(r_1,r_2)}{\log(r_1r_2)}.$$

Extending our notion we can easily define the hyper order (and hyper lower order), generalised order (and generalised lower order) and (p,q)th order (and (p,q)th lower order) of entire functions of two complex variables where p and q are any two positive integers with $p>q$.

Definition 1 *The hyper order $\overline{\rho}(f)$ and hyper lower order $\overline{\lambda}(f)$ of an entire function f of two complex variables are defined as*

$$\overline{\rho}(f)=\limsup_{r_1,r_2\to\infty}\frac{\log^{[3]}F(r_1,r_2)}{\log(r_1r_2)} \text{ and } \overline{\lambda}(f)=\liminf_{r_1,r_2\to\infty}\frac{\log^{[3]}F(r_1,r_2)}{\log(r_1r_2)}.$$

Definition 2 *The generalised order ${}^{(k)}\rho(f)$ and generalised lower order ${}^{(k)}\lambda(f)$ of an entire function f of two complex variables are defined as*

$${}^{(k)}\rho(f)=\limsup_{r_1,r_2\to\infty}\frac{\log^{[k+1]}F(r_1,r_2)}{\log(r_1r_2)} \text{ and } {}^{(k)}\lambda(f)=\liminf_{r_1,r_2\to\infty}\frac{\log^{[k+1]}F(r_1,r_2)}{\log(r_1r_2)}$$

where $k=1,2,3,...$

Definition 3 *The (p,q)th order $\rho_q^p(f)$ and lower (p,q)th order $\lambda_q^p(f)$ of an entire function f of two complex variables are defined as follows*

$$\rho_q^p(f) = \limsup_{r_1,r_2\to\infty} \frac{\log^{[p+1]} F(r_1,r_2)}{\log^{[q]}(r_1r_2)} \text{ and } \lambda_q^p(f) = \liminf_{r_1,r_2\to\infty} \frac{\log^{[p+1]} F(r_1,r_2)}{\log^{[q]}(r_1r_2)}$$

where p, q are any two positive integers with $p > q$. Using the above notion, in this paper we discuss some comparative growth properties of composite entire functions of two complex variables. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [3].

2 Theorems.

In this section we present the main results of the paper.

Theorem 4 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda(fog) \leq \rho(fog) < \infty$ and $0 < \lambda(f) \leq \rho(f) < \infty$. Then for any positive number A,*

$$\frac{\lambda(fog)}{A\rho(f)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \leq \frac{\lambda(fog)}{A\lambda(f)}$$

$$\leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \leq \frac{\rho(fog)}{A\lambda(f)}.$$

Proof. From the definition of $\rho(f)$ and $\lambda(f)$ we have for arbitrary positive ε and for sufficiently large values of r_1, r_2

$$\log^{[2]} FoG(r_1,r_2) \geq [\lambda(fog) - \varepsilon] \log(r_1r_2) \tag{1}$$

$$\text{and } \log^{[2]} F(r_1^A,r_2^A) \leq [A\rho(fog) + \varepsilon] \log(r_1r_2). \tag{2}$$

Now from (1) and (2) it follows for all sufficiently large values of r_1, r_2 that

$$\frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \geq \frac{\lambda(fog) - \varepsilon}{A\rho(fog) + \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \geq \frac{\lambda(fog)}{A\rho(fog)}. \tag{3}$$

Again for a sequence of values of r_1 tending to infinity and for a sequence of values of r_2 tending to infinity,

$$\log^{[2]} FoG(r_1, r_2) \leq [\lambda(fog) + \varepsilon] \log(r_1 r_2) \tag{4}$$

and for all sufficiently large values of r_1, r_2

$$\log^{[2]} F(r_1^A, r_2^A) \geq [A\lambda(f) - \varepsilon] \log(r_1 r_2). \tag{5}$$

Combining (4) and (5) we get for a sequence of values of r_1 tending to infinity and for a sequence of values of r_2 tending to infinity,

$$\frac{\log^{[2]} FoG(r_1, r_2)}{\log^{[2]} F(r_1^A, r_2^A)} \leq \frac{\lambda(fog) + \varepsilon}{A\lambda(f) - \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows that

$$\liminf_{r_1, r_2 \to \infty} \frac{\log^{[2]} FoG(r_1, r_2)}{\log^{[2]} F(r_1^A, r_2^A)} \leq \frac{\lambda(fog)}{A\lambda(f)}. \tag{6}$$

Also for a sequence of values of r_1 tending to infinity and for a sequence of values of r_2 tending to infinity,

$$\log^{[2]} F(r_1^A, r_2^A) \leq [A\lambda(f) + \varepsilon] \log(r_1 r_2). \tag{7}$$

Now from (1) and (7) we obtain for a sequence of values of r_1 tending to infinity and for a sequence of values of r_2 tending to infinity,

$$\frac{\log^{[2]} FoG(r_1, r_2)}{\log^{[2]} F(r_1^A, r_2^A)} \geq \frac{\lambda(fog) - \varepsilon}{A\lambda(f) + \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, we get that

$$\limsup_{r_1, r_2 \to \infty} \frac{\log^{[2]} FoG(r_1, r_2)}{\log^{[2]} F(r_1^A, r_2^A)} \geq \frac{\lambda(fog)}{A\lambda(f)}. \tag{8}$$

Also for all sufficiently large values of r_1, r_2

$$\log^{[2]} FoG(r_1, r_2) \leq [\rho(fog) + \varepsilon] \log(r_1 r_2). \tag{9}$$

From (5) and (9) it follows for all sufficiently large values of r_1, r_2

$$\frac{\log^{[2]} FoG(r_1, r_2)}{\log^{[2]} F(r_1^A, r_2^A)} \leq \frac{\rho(fog) + \varepsilon}{A\lambda(f) - \varepsilon}.$$

Since $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\limsup_{r_1, r_2 \to \infty} \frac{\log^{[2]} FoG(r_1, r_2)}{\log^{[2]} F(r_1^A, r_2^A)} \leq \frac{\rho(fog)}{A\lambda(f)}. \tag{10}$$

Thus the theorem follows from (3), (6), (8) and (10). ■

Theorem 5 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda(fog) \leq \rho(fog) < \infty$ and $0 < \rho(f) < \infty$. Then for any positive number A,*

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \leq \frac{\rho(fog)}{A\rho(f)} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)}.$$

Proof. From the definition of order of an entire function of two variables we get for a sequence of values of r_1 tending to infinity and also for a sequence of values of r_2 tending to infinity,

$$\log^{[2]} F(r_1^A,r_2^A) \geq [A\rho(f) - \varepsilon] \log(r_1 r_2). \tag{11}$$

Now from (9) and (11) it follows for a sequence of values of r_1 tending to infinity and for a sequence of values r_2 tending to infinity,

$$\frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \leq \frac{\rho(fog)+\varepsilon}{A\rho(f)-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \leq \frac{\rho(fog)}{A\rho(f)}. \tag{12}$$

Again for a sequence of values of r_1 tending to infinity and for a sequence of values of r_2 tending to infinity,

$$\log^{[2]} FoG(r_1,r_2) \geq [\rho(fog) - \varepsilon] \log(r_1 r_2). \tag{13}$$

So combining (2) and (13)we get for a sequence of values of r_1 tending to infinity and for a sequence of values of r_2 tending to infinity,

$$\frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \geq \frac{\rho(fog)-\varepsilon}{A\rho(f)+\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \geq \frac{\rho(fog)}{A\rho(f)}. \tag{14}$$

Thus the theorem follows from (12) and (14).The following theorem is a natural consequence of Theorem 1 and Theorem 2. ■

Theorem 6 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda(fog) \leq \rho(fog) < \infty$ and $0 < \lambda(f) \leq \rho(f) < \infty$. Then for any positive number A,*

$$\frac{\lambda(fog)}{A\rho(f)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \leq \min\{\frac{\lambda(fog)}{A\lambda(f)}, \frac{\rho(fog)}{A\rho(f)}\}$$

$$\leq \max\{\frac{\lambda(fog)}{A\lambda(f)}, \frac{\rho(fog)}{A\rho(f)}\} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} F(r_1^A,r_2^A)} \leq \frac{\rho(fog)}{A\lambda(f)}.$$

The proof is omitted.

Remark 7 *If we take $0 < \lambda(g) \leq \rho(g) < \infty$ instead of $0 < \lambda(f) \leq \rho(f) < \infty$ and the other conditions remain the same then Theorem1, Theorem 2 and Theorem 3 are still valid with $G(r_1^A, r_2^A)$ in the denominators of the ratios as we see in the subsequent theorems i.e., Theorem 4, Theorem 5 and Theorem 6 respectively.*

Theorem 8 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda(fog) \leq \rho(fog) < \infty$ and $0 < \lambda(g) \leq \rho(g) < \infty$. Then for any positive number A,*

$$\frac{\lambda(fog)}{A\rho(g)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} G(r_1^A,r_2^A)} \leq \frac{\lambda(fog)}{A\lambda(g)}$$

$$\leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} G(r_1^A,r_2^A)} \leq \frac{\rho(fog)}{A\lambda(g)}.$$

The proof of Theorem 4 is omitted because it can be carried out in the line of Theorem 1. In the line of Theorem 2 we may prove the following theorem.

Theorem 9 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda(fog) \leq \rho(fog) < \infty$ and $0 < \rho(g) < \infty$. Then for any positive number A,*

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} G(r_1^A,r_2^A)} \leq \frac{\rho(fog)}{A\rho(g)} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} G(r_1^A,r_2^A)}.$$

The proof is omitted. The following theorem is a natural consequence of Theorem 4 and Theorem 5.

Theorem 10 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda(fog) \leq \rho(fog) < \infty$ and $0 < \lambda(g) \leq \rho(g) < \infty$. Then for any positive number A,*

$$\frac{\lambda(fog)}{A\rho(g)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} G(r_1^A, r_2^A)} \leq \min\{\frac{\lambda(fog)}{A\lambda(g)}, \frac{\rho(fog)}{A\rho(g)}\}$$

$$\leq \max\{\frac{\lambda(fog)}{A\lambda(g)}, \frac{\rho(fog)}{A\rho(g)}\} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[2]} FoG(r_1,r_2)}{\log^{[2]} G(r_1^A, r_2^A)} \leq \frac{\rho(fog)}{A\lambda(g)}.$$

Using Definition 1, we may obtain the following theorems.

Theorem 11 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \overline{\lambda}(fog) \leq \overline{\rho}(fog) < \infty$ and $0 < \overline{\lambda}(f) \leq \overline{\rho}(f) < \infty$. Then for any positive number A,*

$$\frac{\overline{\lambda}(fog)}{A\overline{\rho}(f)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[3]} FoG(r_1,r_2)}{\log^{[3]} F(r_1^A, r_2^A)} \leq \frac{\overline{\lambda}(fog)}{A\overline{\lambda}(f)}$$

$$\leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[3]} FoG(r_1,r_2)}{\log^{[3]} F(r_1^A, r_2^A)} \leq \frac{\overline{\rho}(fog)}{A\overline{\lambda}(f)}.$$

Theorem 12 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \overline{\lambda}(fog) \leq \overline{\rho}(fog) < \infty$ and $0 < \overline{\rho}(f) < \infty$. Then for any positive number A,*

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[3]} FoG(r_1,r_2)}{\log^{[3]} F(r_1^A, r_2^A)} \leq \frac{\overline{\rho}(fog)}{A\overline{\rho}(f)} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[3]} FoG(r_1,r_2)}{\log^{[3]} F(r_1^A, r_2^A)}.$$

The following theorem is a natural consequence of Theorem 7 and Theorem 8.

Theorem 13 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \overline{\lambda}(fog) \leq \overline{\rho}(fog) < \infty$ and $0 < \overline{\lambda}(f) \leq \overline{\rho}(f) < \infty$. Then for any positive number A,*

$$\frac{\overline{\lambda}(fog)}{A\overline{\rho}(f)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[3]} FoG(r_1,r_2)}{\log^{[3]} F(r_1^A, r_2^A)} \leq \min\{\frac{\overline{\lambda}(fog)}{A\overline{\lambda}(f)}, \frac{\overline{\rho}(fog)}{A\overline{\rho}(f)}\}$$

$$\leq \max\{\frac{\overline{\lambda}(fog)}{A\overline{\lambda}(f)}, \frac{\overline{\rho}(fog)}{A\overline{\rho}(f)}\} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[3]} FoG(r_1,r_2)}{\log^{[3]} F(r_1^A, r_2^A)} \leq \frac{\overline{\rho}(fog)}{A\overline{\lambda}(f)}.$$

Remark 14 *If we consider $0 < \overline{\lambda}(g) \leq \overline{\rho}(g) < \infty$ instead of $0 < \overline{\lambda}(f) \leq \overline{\rho}(f) < \infty$ and the other conditions remain the same then Theorem 7, Theorem 8 and Theorem 9 are still valid with $G(r_1^A, r_2^A)$ in the denominators of the ratios as we see in the subsequent theorems i.e., Theorem 10, Theorem 11 and Theorem 12 respectively.*

Theorem 15 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \overline{\lambda}(fog) \leq \overline{\rho}(fog) < \infty$ and $0 < \overline{\lambda}(g) \leq \overline{\rho}(g) < \infty$. Then for any positive number A,*

$$\frac{\overline{\lambda}(fog)}{A\overline{\rho}(g)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[3]} FoG(r_1,r_2)}{\log^{[3]} G(r_1^A,r_2^A)} \leq \frac{\overline{\lambda}(fog)}{A\overline{\lambda}(g)}$$

$$\leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[3]} FoG(r_1,r_2)}{\log^{[3]} G(r_1^A,r_2^A)} \leq \frac{\overline{\rho}(fog)}{A\overline{\lambda}(g)}.$$

Theorem 16 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \overline{\lambda}(fog) \leq \overline{\rho}(fog) < \infty$ and $0 < \overline{\rho}(g) < \infty$. Then for any positive number A,*

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[3]} FoG(r_1,r_2)}{\log^{[3]} G(r_1^A,r_2^A)} \leq \frac{\overline{\rho}(fog)}{A\overline{\rho}(g)} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[3]} FoG(r_1,r_2)}{\log^{[3]} G(r_1^A,r_2^A)}.$$

The following theorem is a natural consequence of Theorem 10 and Theorem 11.

Theorem 17 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \overline{\lambda}(fog) \leq \overline{\rho}(fog) < \infty$ and $0 < \overline{\lambda}(g) \leq \overline{\rho}(g) < \infty$. Then for any positive number A,*

$$\frac{\overline{\lambda}(fog)}{A\overline{\rho}(g)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[3]} FoG(r_1,r_2)}{\log^{[3]} G(r_1^A,r_2^A)} \leq \min\{\frac{\overline{\lambda}(fog)}{A\overline{\lambda}(g)}, \frac{\overline{\rho}(fog)}{A\overline{\rho}(g)}\}$$

$$\leq \max\{\frac{\overline{\lambda}(fog)}{A\overline{\lambda}(g)}, \frac{\overline{\rho}(fog)}{A\overline{\rho}(g)}\} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[3]} FoG(r_1,r_2)}{\log^{[3]} G(r_1^A,r_2^A)} \leq \frac{\overline{\rho}(fog)}{A\overline{\lambda}(g)}.$$

Using Definition 2, we may obtain the next three theorems.

Theorem 18 *Let f and g be two non-constant entire functions of two complex variables such that $0 < {}^{(k)}\lambda(fog) \leq {}^{(k)}\rho(fog) < \infty$ and $0 < {}^{(k)}\lambda(f) \leq {}^{(k)}\rho(f) < \infty$. Then for any positive number A,*

$$\frac{{}^{(k)}\lambda(fog)}{A^{(k)}\rho(f)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} F(r_1^A,r_2^A)} \leq \frac{{}^{(k)}\lambda(fog)}{A^{(k)}\lambda(f)}$$

$$\leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} F(r_1^A,r_2^A)} \leq \frac{{}^{(k)}\rho(fog)}{A^{(k)}\lambda(f)}$$

where $k = 1, 2, 3, ...$

Theorem 19 *Let f and g be two non-constant entire functions of two complex variables such that $0 < {}^{(k)}\lambda(fog) \le {}^{(k)}\rho(fog) < \infty$ and $0 < {}^{(k)}\rho(f) < \infty$. Then for any positive number A,*

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} F(r_1^A,r_2^A)} \le \frac{{}^{(k)}\rho(fog)}{A^{(k)}\rho(f)} \le \limsup_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} F(r_1^A,r_2^A)}$$

where $k = 1, 2, 3, ...$The following theorem is a natural consequence of Theorem 13 and Theorem 14.

Theorem 20 *Let f and g be two non-constant entire functions of two complex variables such that $0 < {}^{(k)}\lambda(fog) \le {}^{(k)}\rho(fog) < \infty$ and $0 < {}^{(k)}\lambda(f) \le {}^{(k)}\rho(f) < \infty$. Then for any positive number A,*

$$\frac{{}^{(k)}\lambda(fog)}{A^{(k)}\rho(f)} \le \liminf_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} F(r_1^A,r_2^A)} \le \min\{\frac{{}^{(k)}\lambda(fog)}{A^{(k)}\lambda(f)}, \frac{{}^{(k)}\rho(fog)}{A^{(k)}\rho(f)}\}$$

$$\le \max\{\frac{{}^{(k)}\lambda(fog)}{A^{(k)}\lambda(f)}, \frac{{}^{(k)}\rho(fog)}{A^{(k)}\rho(f)}\} \le \limsup_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} F(r_1^A,r_2^A)} \le \frac{{}^{(k)}\rho(fog)}{A^{(k)}\lambda(f)}$$

where $k = 1, 2, 3, ...$

Remark 21 *If we consider $0 < {}^{(k)}\lambda(g) < {}^{(k)}\rho(g) < \infty$ instead of $0 < {}^{(k)}\lambda(f) < {}^{(k)}\rho(f) < \infty$ and the other conditions remain the same then Theorem 13, Theorem 14 and Theorem 15 are still valid with $G(r_1^A, r_2^A)$ in the denominators of the ratios as we see in the subsequent theorems i.e., Theorem 16, Theorem 17 and Theorem 18.*

Theorem 22 *Let f and g be two non-constant entire functions of two complex variables such that $0 < {}^{(k)}\lambda(fog) \le {}^{(k)}\rho(fog) < \infty$ and $0 < {}^{(k)}\lambda(g) \le {}^{(k)}\rho(g) < \infty$. Then for any positive number A,*

$$\frac{{}^{(k)}\lambda(fog)}{A^{(k)}\rho(g)} \le \liminf_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} G(r_1^A,r_2^A)} \le \frac{{}^{(k)}\lambda(fog)}{A^{(k)}\lambda(g)}$$

$$\le \limsup_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} G(r_1^A,r_2^A)} \le \frac{{}^{(k)}\rho(fog)}{A^{(k)}\lambda(g)}$$

where $k = 1, 2, 3, ...$

Theorem 23 *Let f and g be two non-constant entire functions of two complex variables such that $0 < {}^{(k)}\lambda(fog) \le {}^{(k)}\rho(fog) < \infty$ and $0 < {}^{(k)}\rho(g) < \infty$. Then for any positive number A,*

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} G(r_1^A,r_2^A)} \le \frac{{}^{(k)}\rho(fog)}{A^{(k)}\rho(g)} \le \limsup_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} G(r_1^A,r_2^A)}$$

where $k = 1, 2, 3, ...$The following theorem is a natural consequence of Theorem 16 and Theorem 17.

Theorem 24 *Let f and g be two non-constant entire functions of two complex variables such that $0 < {}^{(k)}\lambda(fog) \leq {}^{(k)}\rho(fog) < \infty$ and $0 < {}^{(k)}\lambda(g) \leq {}^{(k)}\rho(g) < \infty$. Then for any positive number A,*

$$\frac{{}^{(k)}\lambda(fog)}{A^{(k)}\rho(g)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} G(r_1^A,r_2^A)} \leq \min\{\frac{{}^{(k)}\lambda(fog)}{A^{(k)}\lambda(g)}, \frac{{}^{(k)}\rho(fog)}{A^{(k)}\rho(g)}\}$$

$$\leq \max\{\frac{{}^{(k)}\lambda(fog)}{A^{(k)}\lambda(g)}, \frac{{}^{(k)}\rho(fog)}{A^{(k)}\rho(g)}\} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[k+1]} FoG(r_1,r_2)}{\log^{[k+1]} G(r_1^A,r_2^A)} \leq \frac{{}^{(k)}\rho(fog)}{A^{(k)}\lambda(g)}$$

where $k = 1, 2, 3, ...$

Theorem 25 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda_q^p(fog) \leq \rho_q^p(fog) < \infty$ and $0 < \lambda_q^m(f) \leq \rho_q^m(f) < \infty$. Then for any positive number A,*

$$\frac{\lambda_q^p(fog)}{A\rho_q^m(f)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} F(r_1^A,r_2^A)} \leq \frac{\lambda_q^p(fog)}{A\lambda_q^m(f)}$$

$$\leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} F(r_1^A,r_2^A)} \leq \frac{\rho_q^p(fog)}{A\lambda_q^m(f)}$$

where p, q and m are any three positive integers such that $p > q$ and $m > q$ i.e., $q < \min\{p, m\}$.

Theorem 26 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda_q^p(fog) \leq \rho_q^p(fog) < \infty$ and $0 < \rho_q^m(f) < \infty$. Then for any positive number A,*

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} F(r_1^A,r_2^A)} \leq \frac{\rho_q^p(fog)}{A\rho_q^m(f)} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} F(r_1^A,r_2^A)}$$

where p, q and m are any three positive integers such that $p > q$ and $m > q$ i.e., $q < \min\{p, m\}$.The following theorem is a natural consequence of Theorem 19 and Theorem 20.

Theorem 27 *Let f and g be two non-constant entire functions of two complex variables such that $0 < \lambda_q^p(fog) \leq \rho_q^p(fog) < \infty$ and $0 < \lambda_q^m(f) \leq \rho_q^m(f) < \infty$. Then for any positive number A,*

$$\frac{\lambda_q^p(fog)}{A\rho_q^m(f)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} F(r_1^A,r_2^A)} \leq \min\{\frac{\lambda_q^p(fog)}{A\lambda_q^m(f)}, \frac{\rho_q^p(fog)}{A\rho_q^m(f)}\}$$

$$\leq \max\{\frac{\lambda_q^p(fog)}{A\lambda_q^m(f)}, \frac{\rho_q^p(fog)}{A\rho_q^m(f)}\} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} F(r_1^A, r_2^A)} \leq \frac{\rho_q^p(fog)}{A\lambda_q^m(f)},$$

where p, q *and* m *are any three positive integers such that* $p > q$ *and* $m > q$ *i.e.,* $q < \min\{p, m\}$.

Remark 28 *If we consider* $0 < \lambda_q^m(g) \leq \rho_q^m(g) < \infty$ *instead of* $0 < \lambda_q^m(f) \leq \rho_q^m(f) < \infty$ *and the other conditions remain the same then Theorem 19, Theorem 20 and Theorem 21 are still valid with* $G(r_1^A, r_2^A)$ *in the denominators of the ratios as we see in the subsequent theorems i.e., Theorem 19, Theorem 20 and Theorem 21.*

Theorem 29 *Let* f *and* g *be two non-constant entire functions of two complex variables such that* $0 < \lambda_q^p(fog) < \infty$ *and* $0 < \lambda_q^m(g) \leq \rho_q^m(g) < \infty$. *Then for any positive number* A,

$$\frac{\lambda_q^p(fog)}{A\rho_q^m(g)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} G(r_1^A, r_2^A)} \leq \frac{\lambda_q^p(fog)}{A\lambda_q^m(g)}$$

$$\leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} G(r_1^A, r_2^A)} \leq \frac{\rho_q^m(fog)}{A\lambda_q^m(g)}$$

where p, q *and* m *are any three positive integers such that* $p > q$ *and* $m > q$ *i.e.,* $q < \min\{p, m\}$.

Theorem 30 *Let* f *and* g *be two non-constant entire functions of two complex variables such that* $0 < \lambda_q^p(fog) \leq \rho_q^p(fog) < \infty$ *and* $0 < \rho_q^m(g) < \infty$. *Then for any positive number* A,

$$\liminf_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} G(r_1^A, r_2^A)} \leq \frac{\rho_q^p(fog)}{A\rho_q^m(g)} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} G(r_1^A, r_2^A)}$$

where p, q *and* m *are any three positive integers such that* $p > q$ *and* $m > q$ *i.e.,* $q < \min\{p, m\}$.

Theorem 31 *Let* f *and* g *be two non-constant entire functions of two complex variables such that* $0 < \lambda_q^p(fog) \leq \rho_q^p(fog) < \infty$ *and* $0 < \lambda_q^m(g) \leq \rho_q^m(g) < \infty$. *Then for any positive number* A,

$$\frac{\lambda_q^p(fog)}{A\rho_q^m(g)} \leq \liminf_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} G(r_1^A, r_2^A)} \leq \min\{\frac{\lambda_q^p(fog)}{A\lambda_q^m(g)}, \frac{\rho_q^p(fog)}{A\rho_q^m(g)}\}$$

$$\leq \max\{\frac{\lambda_q^p(fog)}{A\lambda_q^m(g)}, \frac{\rho_q^p(fog)}{A\rho_q^m(g)}\} \leq \limsup_{r_1,r_2\to\infty} \frac{\log^{[p]} FoG(r_1,r_2)}{\log^{[m]} G(r_1^A, r_2^A)} \leq \frac{\rho_q^p(fog)}{A\lambda_q^m(g)},$$

where p, q *and* m *are any three positive integers such that* $p > q$ *and* $m > q$ *i.e.,* $q < \min\{p, m\}$.

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