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Generalised $L^* - (p, q)$th Order of the Derivative of a Meromorphic Function

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Abstract

In this paper we generalise the results of Datta and Mondal [3].

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1 Introduction, Definitions and Notations.

We know {cf.[10], p.36} that the order of the derivative of an entire function is equal to the order of the function. The same result is proved for a meromorphic function in {cf.[1], [9], [11]}. In [6] and [7] Lahiri proved that the generalised order (generalised lower order) of a meromorphic function f is equal to the generalised order of its derivative f'. Using the notion of (p,q)th order $\big((p,q)$th lower order$\big)$ for any two positive integers with $p > q$ of an entire function introduced by Juneja, Kapoor and Bajpai [5] and the notion of slowly changing functions investigated by Somasundaram and Thamizharasi [8], Datta and Mondal [3] established a relationship between the $L-(p,q)$th order of the derivative of a meromorphic function and that of the original function where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every constant 'a' and p, q are any two positive integers with $p > q$. In this paper we generalise the results of Datta and Mondal [3] and for this we introduce the following definition:

Definition 1. The generalised $L^{*}-(p,q)$th order with rate t, ${}^{(t)}\rho_f^{L^*}(p,q)$ and generalised $L^{*}-(p,q)$th lower order with rate t, ${}^{(t)}\lambda_f^{L^*}(p,q)$ of an entire function f are defined as

$$ {}^{(t)}\rho_f^{L^*}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]}[r \exp^{[t]} L(r)]} $$

$$ \text{and } {}^{(t)}\lambda_f^{L^*}(p,q) = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]}[r \exp^{[t]} L(r)]} $$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, ...$and $\log^{[0]} x = x$ and $\exp^{[t]} x = \exp(\exp^{[t-1]} x)$ for $t = 1, 2, 3, ...$and $\exp^{[0]} x = x$ and also p, q are any two positive integers with $p > q$. When f is meromorphic, one can easily verify that

$$ {}^{(t)}\rho_f^{L^*}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]}[r \exp^{[t]} L(r)]} $$

$$ \text{and } {}^{(t)}\lambda_f^{L^*}(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]}[r \exp^{[t]} L(r)]}. $$

In the paper we do not explain the standard notations and definitions in the theory of entire and meromorphic functions because those are available in [10] and [4].

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [7] Let f be a transcendental meromorphic function. Then

$$ T(r,f') \leq 2T(2r,f) + o\{T(2r,f)\} \text{ for all large values of } r. $$

Lemma 2. { Theorem 4.1, [12]; see also Lemma C, [2]} Let f be a meromorphic function. Then for all large r,

$$T(r,f)<C\{T(2r,f')+\log r\}$$

where C is a constant which is only dependent on $f(0)$.

3 Theorems.

In this section we present the main results of the paper.

Theorem 1. The generalised $L^{*}-(p,q)$th order with rate t of a meromorphic function f is equal to the generalised $L^{*}-(p,q)$th order of its derivative f' where p,q are positive integers and $p>q$ with $t=1,2,3,...$

Proof. We suppose that f is a transcendental meromorphic function because otherwise the theorem follows easily.

From Lemma 1 we get by taking logarithms $(p-1)$times

$$\log^{[p-1]}T(r,f')\leq\log^{[p-1]}T(2r,f)+O(1)$$

which gives that

$$^{(t)}\rho_{f''}^{L^{*}}(p,q)\leq\limsup_{r\to\infty}\{\frac{\log^{[p-1]}T(r,f)}{\log^{[q]}[r\exp^{[t]}L(r)]}.\frac{1}{1-\frac{\log 2}{\log^{[q]}[r\exp^{[t]}L(r)]}}\}$$

$$=\limsup_{r\to\infty}\frac{\log^{[p-1]}T(r,f)}{\log^{[q]}[r\exp^{[t]}L(r)]}.\lim_{r\to\infty}\frac{1}{1-\frac{\log^{[2]}}{\log^{[q]}[r\exp^{[t]}L(r)]}}$$

$$=\ ^{(t)}\rho_{f}^{L^{*}}(p,q). \tag{1}$$

Since f is transcendental, we have

$$\log r=o\{T(r,f)\}.$$

From Lemma 2 we obtain by taking repeated logarithms

$$\log^{[p-1]}T(r,f)+O(1)\leq\log^{[p-1]}T(2r,f')$$

which gives that

$$^{(t)}\rho_{f}^{L^{*}}(p,q)\leq\limsup_{r\to\infty}\frac{\log^{[p-1]}T(r,f')}{\log^{[q]}[r\exp^{[t]}L(r)]}.\lim_{r\to\infty}\frac{1}{1-\frac{\log 2}{\log^{[q]}[r\exp^{[t]}L(r)]}}$$

$$\text{i.e., } {}^{(t)}\rho_f^{L^*}(p,q) \leq {}^{(t)}\rho_{f'}^{L^*}(p,q). \tag{2}$$

Thus the theorem follows from (1) and (2).

Remark 1. Theorem 1 is a generalisation of Theorem 1 [3].

Theorem 2. The generalised $L^* - (p,q)$th lower order with rate t of a meromorphic function f is equal to the generalised $L^* - (p,q)$th lower order of its derivative f' where p,q are positive integers and $p > q$ with $t = 1,2,3,...$

We omit the proof of Theorem 2 as it is similar to that of Theorem 1.

Remark 2. Theorem 2 is a generalisation of Theorem 2 [3].

Theorem 3. If f is a transcendental meromorphic function having a finite number of zeros with

$$f(0) \neq 0, \infty, \; f'(0) \neq 0 \text{ and } {}^{(t)}\rho_f^{L^*}(2,1) < \infty \text{ then } {}^{(t)}\rho_{f'}^{L^*}(p,q) = {}^{(t)}\rho_f^{L^*}(p,q)$$

$$\text{and } {}^{(t)}\lambda_{f'}^{L^*}(p,q) = {}^{(t)}\lambda_f^{L^*}(p,q)$$

where p,q are positive integers and $p > q$ with $t = 1,2,3,...$

Proof. From {Theorem 2.2, [4], p.40} we know that

$$m(r, \frac{f'}{f}) = O(\log r).$$

Also by Theorem {2.3, [4], p.41} we obtain in the present case,

$$\log r = o\{T(r,f)\} \text{ as } r \to \infty.$$

So combining the two we get that

$$m(r, \frac{f'}{f}) = o\{T(r,f)\} \text{ as } r \to \infty.$$

Since f has a finite number of zeros, it is clear that

$$N(r, \frac{1}{f}) = O(\log r).$$

$$\text{Hence } \; N(r, \frac{1}{f}) = o\{T(r,f)\} \text{ as } r \to \infty.$$

$$\text{Now } \; m(r,f') \leq m(r, \frac{f'}{f}) + m(r,f)$$

$$\text{i.e., } m(r,f') \leq m(r,f) + o\{T(r,f)\} \text{ as } r \to \infty.$$

Also if f has a pole of order p at z_0, $f'(z)$ has a pole of order $p+1 \leq 2p$, so that

$$N(r,f') \leq 2N(r,f) \text{ \{p.56,[4]\}}.$$

Thus by addition we deduce that

$$T(r,f') \leq m(r,f) + 2N(r,f) + o\{T(r,f)\}$$

$$\text{i.e., } T(r,f') \leq 2T(r,f) + o\{T(r,f)\}$$

$$\text{i.e., } T(r,f') \leq \{2 + o(1)\}T(r,f) \text{ as } r \to \infty. \tag{3}$$

This gives that

$$^{(t)}\rho_{f'}^{L^*}(p,q) \leq \ ^{(t)}\rho_{f}^{L^*}(p,q). \tag{4}$$

Again we have

$$T(r,f) = m(r,\frac{1}{f}) + N(r,\frac{1}{f}) + O(1)$$

$$\text{i.e., } T(r,f) \leq m(r,\frac{1}{f'}) + m(r,\frac{f'}{f}) + N(r,\frac{1}{f}) + O(1)$$

$$\text{i.e., } T(r,f) \leq m(r,\frac{1}{f'}) + o\{T(r,f)\}$$

$$\text{i.e., } T(r,f) \leq T(r,\frac{1}{f'}) + o\{T(r,f)\}$$

$$\text{i.e., } T(r,f) \leq T(r,f') + o\{T(r,f)\} \text{ as } r \to \infty$$

$$\text{i.e., } \{1 + o(1)\}T(r,f) \leq T(r,f') \text{ as } r \to \infty. \tag{5}$$

This gives that

$$^{(t)}\rho_{f}^{L^*}(p,q) \leq \ ^{(t)}\rho_{f'}^{L^*}(p,q). \tag{6}$$

Thus the first part of the theorem follows from (4) and (6).

$$\text{Similarly, } ^{(t)}\lambda_{f'}^{L^*}(p,q) = \ ^{(t)}\lambda_{f}^{L^*}(p,q).$$

This proves the theorem.

Remark 3. Theorem 3 is a generalisation of Theorem 3 [3].

Remark 4. Theorem 3 can also be proved with a lesser hypothesis

$$N(r,\frac{1}{f}) = O(\log r)$$

than ‘having a finite number of zeros.’

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