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On the Growth of Composition of Entire Functions with Respect to Minimum Modulus

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Abstract

In the paper we study the growth properties of composite entire and meromorphic functions using (p,q)th order which improve some earlier results, where p,q are positive integers and $p > q$.Some results in the form of remarks based upon minimum modulus of entire functions have also been stated in this paper.

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1 Introduction, Definitions and Notations.

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function and g be an entire function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [7] and [3]. In the sequel we use the following notation:

$$\log^{[k]} x = \log(\log^{[k-1]} x) \text{ for } k = 1, 2, 3, ...\text{and } \log^{[0]} x = x.$$

Ther following definition is well known.

Definition 1. The order ρ_f and lower order λ_f of a meromorphic function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire then

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

Juneja, Kapoor and Bajpai [5] defined the (p,q)th order and (p,q)th lower order of an entire function f respectively as follows:

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r} \text{ and } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r}.$$

When f is meromorphic, one can easily verify that

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} r} \text{ and } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} r},$$

where p, q are positive integers and $p > q$.

In this paper we intend to establish some results relating to the growth properties of composite entire and meromorphic functions on the basis of (p,q)th order and minimum modulus of integral (entire) functions improving some previous results where p, q are positive integers and $p > q$.

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [2] If f and g are two entire functions, then for all sufficiently large values of r

$$M(\frac{1}{8}M(\frac{r}{2},g)-|g(0)|\,,f)\leq M(r,fog)\leq M(M(r,g),f).$$

Lemma 2. [6] Let f be entire and g be a transcendental entire function of finite lower order. Then for any $\delta>0$,

$$M(r^{1+\delta},fog)\geq M(M(r,g),f)\ (r\geq r_0).$$

Lemma 3 {[1], [4]}. Let $f(z)$ be an entire function such that

$$\limsup_{r\to\infty}\frac{\log M(r,f)}{(\log r)^2}\leq c<\frac{1}{4e}.$$

If $0<4ec<\delta<1$ then outside a set of upper logarithmic density at most δ,

$$\frac{m(r,f)}{M(r,f)}>k(\delta,c)=\frac{1-2.2\tau}{1+2.2\tau}\text{ where }\tau=\exp\{-\delta/(4ec)\}.$$

If in particular $c=0$ then

$$\frac{m(r,f)}{M(r,f)}\to 1\text{ as }r\to\infty$$

on a set of logarithmic density 1, where

$$m(r,f)=\inf_{|z|=r}|f(z)|\,,\text{ the minimum modulus of }f.$$

3 Theorems.

In this section we present the main results of the paper.

Theorem 1. Let f be entire and g be transcendental entire with $\lambda_g<\infty$.Then

$$\rho_f(p,m)\ \lambda_g(m,q)\ \leq\rho_{fog}(p,q)\ \leq\rho_f(p,m)\rho_g(m,q),$$

where p,q,m,n are positive integers such that $p>m>q$.

Proof. In view of Lemma 2

$$\rho_{fog}(p,q)=\limsup_{r\to\infty}\frac{\log^{[p]}M(r^{1+\delta},fog)}{\log^{[q]}r^{1+\delta}}$$

$$\geq\limsup_{r\to\infty}\frac{\log^{[p]}M(M(r,g),f)}{\log^{[m]}M(r,g)}.\liminf_{r\to\infty}\frac{\log^{[m]}M(r,g)}{\log^{[q]}r}=\rho_f(p,m).\lambda_g(m,q).$$

Again by Lemma 1, $\rho_{fog}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,fog)}{\log^{[q]} r}$

$$\leq \limsup_{r\to\infty} \frac{\log^{[p]} M(M(r,g),f)}{\log^{[m]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[m]} M(r,g)}{\log^{[q]} r} = \rho_f(p,m)\rho_g(m,q).$$

From the above two inequalities we get that

$$\rho_f(p,m)\lambda_g(m,q) \leq \rho_{fog}(p,q) \leq \rho_f(p,m)\rho_g(m,q).$$

This proves the theorem.

Corollary 1. Under the same conditions of Theorem 1,

$$\rho_{fog}(p,q) \geq \lambda_f(p,m)\rho_g(m,q).$$

Proof. By Lemma 2,

$$\rho_{fog}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r^{1+\delta},fog)}{\log^{[q]} r^{1+\delta}}$$

$$\geq \liminf_{r\to\infty} \frac{\log^{[p]} M(M(r,g),f)}{\log^{[m]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[m]} M(r,g)}{\log^{[q]} r} = \lambda_f(p,m).\rho_g(m,q).$$

Thus the corollary is established.

Theorem 2. If f is an entire function and g be transcendental entire with $\lambda_g(m,q) < \infty$. Then

$$\lambda_{fog}(p,q) \geq \lambda_f(p,m)\lambda_g(m,q), \text{ where } p,q,m,n \text{ are positive integers such that } p > q > m.$$

Proof. By Lemma 2

$$\lambda_{fog}(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r^{1+\delta},fog)}{\log^{[q]} r^{1+\delta}}$$

$$\geq \liminf_{r\to\infty} \frac{\log^{[p]} M(M(r,g),f)}{\log^{[m]} M(r,g)} . \liminf_{r\to\infty} \frac{\log^{[m]} M(r,g)}{\log^{[q]} r} = \lambda_f(p,m)\lambda_g(m,q).$$

This proves the theorem.

Remark 1. Under the same hypothesis respectively stated in theorem 1, Corollary 1 and Theorem 2 the conclusions of the theorems can also be drawn by using Lemma 3 on a set of logarithmic density 1.

Remark 2. The second part of Theorem 1 is also valid under the same conditions for meromorphic f and entire g.

Theorem 3. Let f be meromorphic and g be entire such that

$0 < \lambda_{fog}(p,q) \leq \rho_{fog}(p,q) < \infty$ and $0 < \lambda_g(m,q) \leq \rho_g(m,q) < \infty$. Then

$$\frac{\lambda_{fog}(p,q)}{\rho_g(m,q)} \leq \liminf_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \leq \frac{\lambda_{fog}(p,q)}{\lambda_g(m,q)}$$

$$\leq \limsup_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \leq \frac{\rho_{fog}(p,q)}{\lambda_g(m,q)}$$

where p, q, m are positive integers such that $p > q > m$ and $k = 0, 1, 2, ...$

Proof. From the definition of (p,q)th order and (p,q)th lower order we have for arbitrary positive ε and for all large values of r,

$$\log^{[p]} T(r, fog) \geq (\lambda_{fog}(p,q) - \varepsilon) \log^{[q]} r. \tag{1}$$

$$\text{and } \log^{[m]} T(r^A, g^{(k)}) \leq (\rho_g(m,q) + \varepsilon) \log^{[q]} r. \tag{2}$$

Now from (1) and (2) it follows for all large values of r,

$$\frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \geq \frac{\lambda_{fog}(p,q) - \varepsilon}{(\rho_g(m,q) + \varepsilon)}. \tag{3}$$

As $\varepsilon(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \geq \frac{\lambda_{fog}(p,q)}{\rho_g(m,q)}.$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]} T(r, fog) \leq (\lambda_{fog}(p,q) + \varepsilon) \log^{[q]} r \tag{4}$$

and for all large values of r,

$$\log^{[m]} T(r^A, g^{(k)}) \geq (\lambda_g(m,q) - \varepsilon) \log^{[q]} r. \tag{5}$$

So combining (4) and (5) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \leq \frac{\lambda_{fog}(p,q) + \varepsilon}{(\lambda_g(m,q) - \varepsilon)}.$$

Since $\varepsilon(>0)$ is arbitrary it follows that

$$\liminf_{r\to\infty}\frac{\log^{[p]}T(r,fog)}{\log^{[m]}T(r^A,g^{(k)})}\le\frac{\lambda_{fog}(p,q)}{\lambda_g(m,q)}. \tag{6}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[m]}T(r^A,g^{(k)})\le(\lambda_g(m,q)+\varepsilon)\log^{[q]}r. \tag{7}$$

Now from (1) and (7) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]}T(r,fog)}{\log^{[m]}T(r^A,g^{(k)})}\ge\frac{\lambda_{fog}(p,q)-\varepsilon}{(\lambda_g(m,q)+\varepsilon)}.$$

Choosing $\varepsilon\to 0$ we get that

$$\limsup_{r\to\infty}\frac{\log^{[p]}T(r,fog)}{\log^{[m]}T(r^A,g^{(k)})}\ge\frac{\lambda_{fog}(p,q)}{\lambda_g(m,q)}. \tag{8}$$

Also for all large values of r,

$$\log^{[p]}T(r,fog)\le(\rho_{fog}(p,q)+\varepsilon)\log^{[q]}r. \tag{9}$$

So from (5) and (9) it follows for all large values of r,

$$\frac{\log^{[p]}T(r,fog)}{\log^{[m]}T(r^A,g^{(k)})}\le\frac{\rho_{fog}(p,q)+\varepsilon}{A(\lambda_g(m,q)-\varepsilon)}$$

As $\varepsilon(>0)$ is arbitrary we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[p]}T(r,fog)}{\log^{[m]}T(r^A,g^{(k)})}\le\frac{\rho_{fog}(p,q)}{A\lambda_g(m,q)}. \tag{10}$$

Thus the theorem follows from (3), (6), (8) and (10).

Theorem 4. Let f be meromorphic and g be entire such that

$0<\lambda_{fog}(p,q)\le\rho_{fog}(p,q)<\infty$ and $0<\rho_g(m,q)<\infty$. Then

$$\liminf_{r\to\infty}\frac{\log^{[p]}T(r,fog)}{\log^{[m]}T(r,g^{(k)})}\le\frac{\rho_{fog}(p,q)}{\rho_g(m,q)}\le\limsup_{r\to\infty}\frac{\log^{[p]}T(r,fog)}{\log^{[m]}T(r,g^{(k)})}$$

where p,q,m are positive integers such that $p>q>m$ and $k=0,1,2,...$

Proof. From the definition of (p,q)th order we get for a sequence of values of r tending to infinity,

$$\log^{[m]} T(r^A, g^{(k)}) \geq (\rho_g(m,q) - \varepsilon) \log^{[q]} r. \tag{11}$$

Now from (9) and (11) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \leq \frac{\rho_{fog}(p,q) + \varepsilon}{A(\rho_g(m,q) - \varepsilon)}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \leq \frac{\rho_{fog}(p,q)}{A\rho_g(m,q)}. \tag{12}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[p]} T(r, fog) \geq (\rho_{fog}(p,q) - \varepsilon) \log^{[q]} r. \tag{13}$$

So combining (2) and (13) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \geq \frac{\rho_{fog}(p,q) - \varepsilon}{A(\rho_g(m,q) + \varepsilon)}.$$

Since $\varepsilon(> 0)$ is arbitrary it follows that

$$\limsup_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \geq \frac{\rho_{fog}(p,q)}{A\rho_g(m,q)}. \tag{14}$$

Thus the theorem follows from (12) and (14).

In view of Theorem 1, Theorem 2 and Theorem 3 we may state the following theorem without proof.

Theorem 5. Let f be meromorphic and g be entire such that

$$0 < \lambda_{fog}(p,q) \leq \rho_{fog}(p,q) < \infty \text{ and } 0 < \lambda_g(m,q) \leq \rho_g(m,q) < \infty.$$

Then for any positive number A,

$$\frac{\lambda_f(p,m)\lambda_g(m,q)}{A\rho_g(m,q)} \leq \frac{\lambda_{fog}(p,q)}{A\rho_g(m,q)} \leq \liminf_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \leq \frac{\lambda_{fog}(p,q)}{A\lambda_g(m,q)}$$

$$\leq \limsup_{r\to\infty} \frac{\log^{[p]} T(r, fog)}{\log^{[m]} T(r^A, g^{(k)})} \leq \frac{\rho_{fog}(p,q)}{A\lambda_g(m,q)} \leq \frac{\rho_f(p,m)\rho_g(m,q)}{A\lambda_g(m,q)},$$

where p, q, m are positive integers such that $p > q > m$ and $k = 0, 1, 2, ...$

The following theorem is a natural consequence of Theorem 3 and Theorem 4.

Theorem 6. Let f be meromorphic and g be entire such that

$$0 < \lambda_{fog}(p,q) \leq \rho_{fog}(p,q) < \infty \text{ and } 0 < \lambda_g(m,q) \leq \rho_g(m,q) < \infty.$$

Then for any positive number A,

$$\liminf_{r\to\infty} \frac{\log^{[p]} T(r,fog)}{\log^{[m]} T(r^A, g^{(k)})} \leq \min\{\frac{\lambda_{fog}(p,q)}{A\lambda_g(m,q)}, \frac{\rho_{fog}(p,q)}{A\rho_g(m,q)}\}$$

$$\leq \max\{\frac{\lambda_{fog}(p,q)}{A\lambda_g(m,q)}, \frac{\rho_{fog}(p,q)}{A\rho_g(m,q)}\} \leq \limsup_{r\to\infty} \frac{\log^{[p]} T(r,fog)}{\log^{[m]} T(r^A, g^{(k)})},$$

where p, q, m are positive integers such that $p > q > m$ and $k = 0, 1, 2, ...$

The proof is omitted.

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