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Estimation of Relative Order of Entire and Meromorphic Functions in Terms of Slowly Changing Functions

Sanjib Kumar Datta

Department of Mathematics, University of Kalyani
P.O.-Kalyani, Dist-Nadia, Pin-741235, West Bengal, India
Former Address:
(Department of Mathematics, University of North Bengal
P.O.-North Bengal University, Raja Rammohunpur
Dist-Darjeeling, Pin-734013, West Bengal, India)
sanjib_kr_datta@yahoo.co.in
s_kr_datta_ku@yahoo.co.in
sk_datta_nbu@yahoo.co.in

Arkojyoti Biswas

Ranaghat Yusuf Institution
P.O.-Ranaghat, Dist-Nadia, PIN-741201, West Bengal, India
arkojyotibiswas@gmail.com

Abstract

In the paper we introduce the definition of *relative* $L-order$ of entire and meromorphic functions and obtain its integral representation.We also show that the relative $L-order$ of a meromorphic function is the same as that of its derivative.

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1 Introduction, Definitions and Notations.

Let f be meromorphic and g be entire in the open complex plane $\mathbb{C}$.Lahiri and Banerjee [5] introduced the definition of relative order of f with respect to g.Their definitions are as follows:

Definition 1 ([5]) *The relative order of f with respect to g denoted by $\rho_g(f)$ is defined as follows:*

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \ \ for\ all\ large\ r\ \} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.\end{aligned}$$

Definition 2 ([1]) *An entire function g is said to have the property (A) if for any $\sigma > 1$ and for all large r,*

$$\{G(r)\}^2 < G(r^\sigma) \ where\ G(r) = \max\{|g(z)| : |z| = r\}.$$

Definition 3 *Two entire functions g_1 and g_2 are said to be asymptotically equivalent if there exists l $(0 < l < \infty)$ such that*

$$\frac{G_1(r)}{G_2(r)} \to l \ as\ r \to \infty$$

and in that case we write $g_1 \sim g_2$ .Clearly if $g_1 \sim g_2$ then $g_2 \sim g_1$.

Let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.,* $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. *Singh and Barker* [6] defined it in the following way:

Definition 4 ([6]) *A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon\,(> 0)$,*

$$\frac{1}{k^\varepsilon} \le \frac{L(kr)}{L(r)} \le k^\varepsilon \ for\ r \ge r(\varepsilon)\ and$$

uniformly for $k\,(\ge 1)$.

If further, $L(r)$ is differentiable, the above condition is equivalent to

$$\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0\ .$$

In this paper we introduce the definition of relative $L-order$ of a meromorphic function with respect to an entire function and obtain its integral representation.We also study its asymptotic behaviour and show that the relative $L-order$ of a meromorphic function is the same as that of its derivative.We do not explain the standard definitions and notations in the theory of entire and meromorphic functions as those are available in [7] and [3].

We now introduce the following definition:

Definition 5 *Let f be meromorphic and g be entire.The relative $L-order$ of f with respect to g denoted by $\rho_g^L(f)$ is defined as*

$$\rho_g^L(f) = \inf\{\mu > 0 : T_f(r) < T_g[(rL(r))^\mu] \ \ for\ all\ large\ r\ \}.$$

If $L(r) \equiv 1$, Definition 5 coincides with the definition of the relative order of f with respect to g.

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 ([1]) *Let g be entire and $\alpha > 1, 0 < \beta < \alpha$. Then*

$$G(\alpha r) > \beta\ G(r) \text{ for all large } r.$$

Lemma 2 ([1]) *Let g be entire which has property(A). Then for any positive integer n and for all $\sigma > 1$*

$$\{G(r)\}^n < G(r^\sigma) \text{ holds for all large } r.$$

Lemma 3 *Let g be entire. Then*

$$T_g(r) \leq \log G(r) \leq 3\ T_g(2r) \text{ for all large } r.$$

Lemma 3 follows from Theorem 1.6 {cf.p 18, [3] } on putting $R = 2r$.

Lemma 4 ([4]) *Let f be a transcendental meromorphic function . Then*

$$T_{f'}(r) \leq 2T_f(2r) + O\{T_f(2r)\} \text{ for all large } r.$$

Lemma 5 ([2, 8]) *Let f be a meromorphic function . Then*

$$T_f(r) < C\{T_{f'}(2r) + \log r\}$$

where C is a constant which depends only on $f(0)$.

Lemma 6 *Let $\int_{r_0}^{\infty} \frac{T_g^{-1}T_f(r)}{[rL(r)]^{k+1}}dr$ $(r_0 > 0)$ converges for $0 < k < \infty$. Then*

$$\limsup_{r\to\infty} \frac{T_g^{-1}T_f(r)}{[rL(r)]^k} = 0.$$

Proof. Since $\int_{r_0}^{\infty} \frac{T_g^{-1}T_f(r)}{[rL(r)]^{k+1}}dr$ $(r_0 > 0)$ converges for $0 < k < \infty$, for given $\varepsilon(> 0)$ there exists $R = R(\varepsilon)$ such that for $r \geq R(\varepsilon) > r_0 > 0$ we have

$$\int_{r}^{r+rL(r)} \frac{T_g^{-1}T_f(t)}{[tL(t)]^{k+1}}dt < \varepsilon.$$

Since $T_g^{-1}T_f(t)$ is an increasing function, we have

$$\frac{T_g^{-1}T_f(r)}{[rL(r)]^{k+1}}\int_r^{r+rL(r)} dt < \int_r^{r+rL(r)} \frac{T_g^{-1}T_f(t)}{[tL(t)]^{k+1}}dt < \varepsilon$$

i.e., for $r \geq R(\varepsilon)$ we have

$$\frac{T_g^{-1}T_f(r)}{[rL(r)]^{k}} < \varepsilon$$

$$i.e., \limsup_{r\to\infty} \frac{T_g^{-1}T_f(r)}{[rL(r)]^{k}} = 0.$$

■

3 Theorems.

In this section we present the main results of the paper.

Theorem 1 *Let f be meromorphic and g be entire.Then*

$$\rho_g^L(f) = \limsup_{r\to\infty} \frac{\log T_g^{-1}T_f(r)}{\log[rL(r)]}.$$

Proof of Theorem 1 follows from Definition 5 and so is omitted.

Theorem 2 *Let f be meromorphic, g be entire and $\rho_g^L(f)$ be the relative $L-$ order of f with respect to g.Then $\rho_g^L(f) = \inf k$ such that the integral $\int_{r_0}^{\infty} \frac{T_g^{-1}T_f(r)}{[rL(r)]^{k+1}}dr$ $(r_0 > 0)$ converges.*

Proof. Case 1 :When $\rho_g^L(f) = \inf k$ is finite.

As $\rho_g^L(f)$ is finite then for arbitrary $\varepsilon(> 0)$ the integral $\int_{r_0}^{\infty} \frac{T_g^{-1}T_f(r)}{[rL(r)]^{\rho_g^L(f)+\varepsilon+1}}dr$ converges.Therefore by *Lemma* 6 we get that

$$\limsup_{r\to\infty} \frac{T_g^{-1}T_f(r)}{[rL(r)]^{\rho_g^L(f)+\varepsilon}} = 0$$

i.e., for all sufficiently large values of r we obtain

$$\frac{T_g^{-1}T_f(r)}{[rL(r)]^{\rho_g^L(f)+\varepsilon}} < \varepsilon_1$$

$$i.e., \ \frac{\log T_g^{-1}T_f(r)}{\log[rL(r)]} < \frac{\log \varepsilon_1}{\log[rL(r)]} + \rho_g^L(f) + \varepsilon.$$

As $\varepsilon(>0)$ is arbitrary we obtain

$$\limsup_{r\to\infty} \frac{\log T_g^{-1}T_f(r)}{\log[rL(r)]} \leq \rho_g^L(f). \tag{1}$$

On the other hand divergence of the integral $\int\limits_{r_0}^{\infty} \frac{T_g^{-1}T_f(r)}{[rL(r)]^{\rho_g^L(f)-\varepsilon+1}}dr$ implies that there exists a sequence of values of r tending to infinity such that

$$T_g^{-1}T_f(r) > [rL(r)]^{\rho_g^L(f)-2\varepsilon}$$

$$i.e., \frac{\log T_g^{-1}T_f(r)}{\log[rL(r)]} > \rho_g^L(f) - 2\varepsilon.$$

As $\varepsilon(>0)$ is arbitrary it follows that

$$\limsup_{r\to\infty} \frac{\log T_g^{-1}T_f(r)}{\log[rL(r)]} \geq \rho_g^L(f). \tag{2}$$

From (1) and (2) we get that

$$\limsup_{r\to\infty} \frac{\log T_g^{-1}T_f(r)}{\log[rL(r)]} = \rho_g^L(f).$$

Now from the definition of $\rho_g^L(f)$ we obtain for arbitrary $\varepsilon(>0)$ and for all sufficiently large values of r that

$$T_g^{-1}T_f(r) < [rL(r)]^{\rho_g^L(f)+\varepsilon}. \tag{3}$$

Also for a sequence of values of r tending to infinity we get

$$T_g^{-1}T_f(r) > [rL(r)]^{\rho_g^L(f)-\varepsilon}. \tag{4}$$

We set $k = \rho_g^L(f) + 2\varepsilon$ and see that for all sufficiently large values of r

$$\frac{T_g^{-1}T_f(r)}{[rL(r)]^{k+1}} < \frac{1}{[rL(r)]^{\varepsilon+1}}$$

and so the integral $\int\limits_{r_0}^{\infty} \frac{T_g^{-1}T_f(r)}{[rL(r)]^{k+1}}dr$ converges.Clearly $\inf k \leq \rho_g^L(f)$.

Similarly using (4) we can show that $\inf k \geq \rho_g^L(f)$.*So* in this case $\inf k = \rho_g^L(f)$.

Also it is clear that if $\rho_g^L(f) = 0$ *then* $\inf k = 0$ and conversely.

Finally let $\inf k = \infty$.Then for any positive number G the integral $\int\limits_{r_0}^{\infty} \frac{T_g^{-1}T_f(r)}{[rL(r)]^{G+1}}dr$ diverges.So there exists a sequence of values of r tending to infinity such that for arbitrary $\varepsilon(> 0)$ we get

$$T_g^{-1}T_f(r) > [rL(r)]^{G-\varepsilon}$$

$$i.e., \ \limsup_{r\to\infty} \frac{\log T_g^{-1}T_f(r)}{\log[rL(r)]} \geq G - \varepsilon.$$

As $\varepsilon(> 0)$ is arbitrary and G is any positive number we get

$$\limsup_{r\to\infty} \frac{\log T_g^{-1}T_f(r)}{\log[rL(r)]} = \infty.$$

Next suppose that

$$\limsup_{r\to\infty} \frac{\log T_g^{-1}T_f(r)}{\log[rL(r)]} = \infty.$$

Then for any finite $G(> 0)$ we obtain for a sequence of values of r tending to infinity that

$$\frac{\log T_g^{-1}T_f(r)}{\log[rL(r)]} > G$$

$$i.e., \ T_g^{-1}T_f(r) > [rL(r)]^G. \tag{5}$$

If possible let the integral $\int\limits_{r_0}^{\infty} \frac{T_g^{-1}T_f(r)}{[rL(r)]^{G+1}}dr$ converges.Then by Lemma 6 we obtain that

$$\limsup_{r\to\infty} \frac{T_g^{-1}T_f(r)}{[rL(r)]^G} = 0.$$

So for all sufficiantly large values of r it follows that

$$T_g^{-1}T_f(r) < [rL(r)]^G$$

which contradicts (5) .This shows that the integral $\int\limits_{r_0}^{\infty} \frac{T_g^{-1}T_f(r)}{[rL(r)]^{G+1}}dr$ diverges for any finite $G.Hence$ $\inf k = \infty.This$ shows that for any case $\inf k = \rho_g^L(f)$. This completes the proof. ■

Theorem 3 *Let g be entire and f be meromorphic with relative $L-$ order $\rho_g^L(f)$ with respect to g and $\varepsilon(>0)$ be arbitrary.Then*

$$T_f(r) = O(\log G([rL(r)]^{\rho_g^L(f)+\varepsilon})) \text{ holds for all large } r.$$

Conversely, if for a meromorphic function f and entire g with property (A)

$$T_f(r) = O(\log G([rL(r)]^{k+\varepsilon})) \text{ holds for all large } r$$

and

$$T_f(r) = O(\log G([rL(r)]^{k-\varepsilon})) \text{ does not hold for all large values of } r,$$

then $k = \rho_g^L(f)$.

Proof. From definition we get for arbitrary $\varepsilon(>0)$ and for all sufficiently large values of r that

$$T_f(r) < T_g([rL(r)]^{\rho_g^L(f)+\varepsilon}).$$

From *Lemma* 3 we get that

$$\begin{aligned} T_f(r) &< \log G([rL(r)]^{\rho_g^L(f)+\varepsilon}) \\ i.e., T_f(r) &= O(\log G([rL(r)]^{\rho_g^L(f)+\varepsilon})). \end{aligned}$$

Conversely, if $T_f(r) = O(\log G([rL(r)]^{k+\varepsilon}))$ then for all sufficiently large values of r and for $\alpha > 1$ we obtain that

$$\begin{aligned} T_f(r) &< [\alpha]\log G([rL(r)]^{k+\varepsilon}) \\ &= \frac{1}{3}\log G([rL(r)]^{k+\varepsilon})^{3[\alpha]}. \end{aligned}$$

Now from *Lemma* 2 and *Lemma* 3 we get for any $\sigma > 1$ that

$$T_f(r) \le \frac{1}{3}\log G([rL(r)]^{(k+\varepsilon)\sigma})$$

$$\textit{and so} \quad T_f(r) < T_g([2rL(2r)]^{(k+\varepsilon)\sigma}).$$

As $L(2r) \sim L(r)$ we get

$$\limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log[rL(r)]} \leq (k+\varepsilon)\sigma.$$

Since $\varepsilon(>0)$ is arbitrary, letting $\sigma \to 1+$ it follows that

$$\limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log[rL(r)]} \leq k. \tag{6}$$

Again there exists a sequence of values of r tending to infinity such that

$$T_f(r) > \log G([rL(r)]^{k-\varepsilon}).$$

So from *Lemma* 3 we get for a sequence of values of r tending to infinity that

$$T_f(r) \geq T_g([rL(r)]^{k-\varepsilon}).$$

As $\varepsilon(>0)$ is arbitrary it follows that

$$\limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log[rL(r)]} \geq k. \tag{7}$$

Combining (6) and (7) we get $k = \rho_g^L(f). This$ completes the proof. ■

Theorem 4 *Let g be entire with property (A) and f_1 and f_2 be meromorphic with relative $L-order$ $\rho_g^L(f_1)$ and $\rho_g^L(f_2)$ with respect to g respectively.Then*

$$(i)\ \rho_g^L(f_1 \pm f_2) \leq \max\{\rho_g^L(f_1), \rho_g^L(f_2)\}$$

and

$$\begin{aligned}(ii)\quad \rho_g^L(f_1.f_2) &\leq \max\{\rho_g^L(f_1), \rho_g^L(f_2)\} \\ \textit{and } \rho_g^L(\frac{f_1}{f_2}) &\leq \max\{\rho_g^L(f_1), \rho_g^L(f_2)\}.\end{aligned}$$

The equality holds in (ii) if $\rho_g^L(f_1) \neq \rho_g^L(f_2)$.

Proof. (i) We may assume that both $\rho_g^L(f_1)$ *and* $\rho_g^L(f_2)$ are finite otherwise the theorem is obvious.Let $\rho = \max\{\rho_g^L(f_1), \rho_g^L(f_2)\}$.Then for arbitrary $\varepsilon(>0)$ and for all large values of r we get by *Lemma* 3 that

$$\begin{aligned} T_{f_1}(r) &< T_g([rL(r)]^{\rho_g^L(f_1)+\varepsilon}) \leq \log G([rL(r)]^{\rho_g^L(f_1)+\varepsilon}) \\ \textit{and} \quad T_{f_2}(r) &< T_g([rL(r)]^{\rho_g^L(f_2)+\varepsilon}) \leq \log G([rL(r)]^{\rho_g^L(f_2)+\varepsilon}).\end{aligned}$$

Now for all large values of r we obtain

$$\begin{aligned} T_{f_1 \pm f_2}(r) &\leq T_{f_1}(r) + T_{f_2}(r) + O(1) \\ &< \log G([rL(r)]^{\rho_g^L(f_1)+\varepsilon}) + \log G([rL(r)]^{\rho_g^L(f_2)+\varepsilon}) + O(1) \\ &< \log G([rL(r)]^{\rho+\varepsilon}) + \log G([rL(r)]^{\rho+\varepsilon}) + \log G([rL(r)]^{\rho+\varepsilon}) \\ &\leq \frac{1}{3} \log G([rL(r)]^{\rho+\varepsilon})^9. \end{aligned}$$

Now from *Lemma* 2 we get for any $\sigma > 1$ that

$$T_{f_1 \pm f_2}(r) \leq \frac{1}{3} \log G([rL(r)]^{(\rho+\varepsilon)\sigma}).$$

By *Lemma* 3 it follows that

$$\begin{aligned} T_{f_1 \pm f_2}(r) &\leq T_g([2rL(2r)]^{(\rho+\varepsilon)\sigma}) \\ i.e., \frac{\log T_g^{-1} T_{f_1 \pm f_2}(r)}{\log[rL(r)]} &\leq [\frac{\log 2}{\log[rL(r)]} + \frac{\log[rL(2r)]}{\log[rL(r)]}](\rho+\varepsilon)\sigma\ . \end{aligned}$$

As $L(2r) \sim L(r)$ we get

$$\limsup_{r \to \infty} \frac{\log T_g^{-1} T_{f_1 \pm f_2}(r)}{\log[rL(r)]} \leq (\rho+\varepsilon)\sigma.$$

As $\varepsilon(>0)$ is arbitrary, letting $\sigma \to 1+$ we get $\rho_g^L(f_1 \pm f_2) \leq \rho, which$ proves the first part of Theorem 4.

(ii) Since $T_{f_1 f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r)$, we obtain as above that $\rho_g^L(f_1.f_2) \leq \rho$.

Also $\frac{f_1}{f_2} = f_1.\frac{1}{f_2}$.Therefore $T_{\frac{f_1}{f_2}}(r) \leq T_{f_1}(r) + T_{f_2}(r) + O(1)$ and so $\rho_g^L(\frac{f_1}{f_2}) \leq \rho$.

Now let $f = f_1.f_2$ and $\rho_g^L(f_1) < \rho_g^L(f_2)$. Then $\rho_g^L(f) \leq \rho_g^L(f_2)$.

Also since $f_2 = \frac{f}{f_1}$,applying first part of (ii) we get $\rho_g^L(f_2) \leq \max\{\rho_g^L(f_1), \rho_g^L(f)\}$.

As $\rho_g^L(f_1) < \rho_g^L(f_2)$ we see that $\rho_g^L(f_2) < \rho_g^L(f)$ and hence $\rho_g^L(f_2) = \rho_g^L(f) = \max\{\rho_g^L(f_1), \rho_g^L(f_2)\}$ where $\rho_g^L(f_1) \neq \rho_g^L(f_2)$.This completes the proof. ■

Theorem 5 *Let g_1 and g_2 be two entire functions with property (A) and $g_1 \sim g_2$ .If f be meromorphic then $\rho_{g_1}^L(f) = \rho_{g_2}^L(f)$.*

Proof. Let $\varepsilon(>0)$ be arbitrary.By *Lemma* 1 we get for $\alpha > 1$ such that $l + \varepsilon < \alpha$ and for all large r that

$$G_1(r) < (l+\varepsilon)G_2(r) < G_2(\alpha r). \tag{8}$$

Now from *Lemma* 3 we obtain

$$\begin{aligned} T_f(r) &< T_{g_1}([rL(r)]^{\rho_{g_1}^L(f)+\varepsilon}) \\ &\leq \log G_1([rL(r)]^{\rho_{g_1}^L(f)+\varepsilon}). \end{aligned}$$

By (8) we get

$$\begin{aligned} T_f(r) &\leq \log G_2(\alpha[rL(r)]^{\rho^L_{g_1}(f)+\varepsilon}) \\ &= \frac{1}{3}\log G_2(\alpha[rL(r)]^{\rho^L_{g_1}(f)+\varepsilon})^3. \end{aligned}$$

By *Lemma* 2 we get for any $\sigma > 1$ that

$$T_f(r) \leq \frac{1}{3}\log G_2[(\alpha[rL(r)]^{\rho^L_{g_1}(f)+\varepsilon})^\sigma].$$

From *Lemma* 3 we obtain

$$\begin{aligned} T_f(r) &\leq T_{g_2}[(\alpha[2rL(2r)]^{\rho^L_{g_1}(f)+\varepsilon})^\sigma] \\ i.e., \frac{\log T_{g_2}^{-1}T_f(r)}{\log[rL(r)]} &\leq \frac{\sigma[\log \alpha(2)^{\rho^L_{g_1}(f)+\varepsilon}]}{\log[rL(r)]} + [(\rho^L_{g_1}(f)+\varepsilon)\sigma]\frac{\log[rL(2r)]}{\log[rL(r)]}. \end{aligned}$$

As $L(2r) \sim L(r)$ and $\varepsilon(>0)$ is arbitrary, letting $\sigma \to 1+$ we get $\rho^L_{g_1}(f) \leq \rho^L_{g_2}(f)$.

Since also $g_2 \sim g_1$,we obtain $\rho^L_{g_2}(f) \leq \rho^L_{g_1}(f)$ and this completes the proof. ■

Remark 1 The converse of Theorem 5 is not always true which is evident from the following example.

Example 1 Let $g_1(z) = \exp z$ and $g_2(z) = \exp(2z)$ so that $G_1(z) = e^r$ and $G_2(z) = e^{2r}$ .Now $\frac{G_1(z)}{G_2(z)} \to 0$ *as* $r \to \infty$ and so $g_1 \nsim g_2$ .But

$$\begin{aligned} \rho^L_{g_2}(f) &= \limsup_{r\to\infty} \frac{\log T_{g_2}^{-1}T_f(r)}{\log[rL(r)]} \\ &= \limsup_{r\to\infty} \frac{\log[\frac{\pi}{2}T_f(r)]}{\log[rL(r)]} \\ &= \limsup_{r\to\infty} \frac{\log[\pi T_f(r)]}{\log[rL(r)]} = \rho^L_{g_1}(f). \blacksquare \end{aligned}$$

Theorem 6 *Let f be transcendental meromorphic and g be entire with property (A).Then $\rho^L_g(f) = \rho^L_g(f')$.*

Proof. From *Lemma* 4 and *Lemma* 5 we get for all large values of r and for k and k$'$ > 1 that

$$T_{f'}(r) \leq [\mathrm{k}]\, T_f(2r) \tag{9}$$

$$and\ T_f(r) \leq [\mathrm{k}']\, T_{f'}(2r). \tag{10}$$

Now from definition we get for arbitrary $\varepsilon(>0)$ and for all large values of r that

$$T_f(2r) \leq T_g([2rL(2r)]^{(\rho_g^L(f)+\varepsilon)}).$$

Now from (9) we get for all large values of r that

$$T_{f'}(r) \leq [\mathrm{k}]\, T_g([2rL(2r)]^{(\rho_g^L(f)+\varepsilon)}).$$

From *Lemma* 3 we obtain

$$\begin{aligned} T_{f'}(r) &< [\mathrm{k}] \log G([2rL(2r)]^{(\rho_g^L(f)+\varepsilon)}) \\ &< \frac{1}{3} \log[G([2rL(2r)]^{(\rho_g^L(f)+\varepsilon)})]^{3[\mathrm{k}]}. \end{aligned}$$

Now from *Lemma* 2 we get for any $\sigma > 1$ that

$$T_{f'}(r) \leq \frac{1}{3} \log G([2rL(2r)]^{(\rho_g^L(f)+\varepsilon)\sigma}).$$

Also from *Lemma* 3 it follows that

$$\begin{aligned} T_{f'}(r) &\leq T_g(2[2rL(2r)]^{(\rho_g^L(f)+\varepsilon)\sigma}) \\ i.e., \frac{\log T_g^{-1} T_{f'}(r)}{\log[rL(r)]} &\leq \frac{\log 2^{(\rho_g^L(f)+\varepsilon)\sigma+1}}{\log[rL(r)]} + \frac{(\rho_g^L(f)+\varepsilon)\ \sigma \log[rL(2r)]}{\log[rL(r)]}. \end{aligned}$$

As $L(2r) \sim L(r)$ and $\varepsilon(>0)$ is arbitrary, letting $\sigma \to 1+$ we get

$$\begin{aligned} \limsup_{r\to\infty} \frac{\log T_g^{-1} T_{f'}(r)}{\log[rL(r)]} &\leq \rho_g^L(f) \\ i.e., \quad \rho_g^L(f') &\leq \rho_g^L(f). \end{aligned}$$

Using (10) we obtain similarly $\rho_g^L(f) \leq \rho_g^L(f').This$ completes the proof. ■

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