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Infinite Boundary Value Problems for Impulsive Differential Equations in Banach Spaces*

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Abstract

By employing the method of upper and lower solutions and the monotone iterative technique, the author studies a kind of infinite boundary value problems for first order impulsive differential equations with infinite skip points in Banach spaces, the existence of the maximal and minimal solutions are obtained under weak conditions, which improves and extends some known results.

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1. Introduction

The theory of impulsive differential equations is one of the new and important branch of differential equations and it has extensive physical background and modern mathematical model. It deals with the rapid changes or skips of development process in a fixed time or unfixed time, and it is a more directly

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reflection of the natural development process. Thus, it becomes a very important field in recent years $[1, 4-6, 8]$. This article deals with the existence of solutions to infinite boundary value problems of impulsive differential equation in a Banach space E

$$\begin{cases} u'(t) = \ f(t, u(t)), & t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), & k = 1, 2, \cdots, \\ u(\infty) = \beta u(0), \end{cases} \tag{1}$$

where $J = [0, +\infty),\ f \in C(J \times E,\ E);\ 0 < t_1 < t_2 < \cdots < t_k < \cdots,\ t_k \to \infty (k \to \infty);\ I_k \in C(E, E)$ is an impulsive function, $k = 1, 2, \cdots, u(\infty) = \lim\limits_{t\to\infty} u(t),\ \beta \geq 1$. $\Delta u|_{t=t_k}$denotes the jump of $u(t)$ at $t = t_k$, i.e. $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^-)$ and $u(t_k^+)$ represent the right and left limits of $u(t)$ at $t = t_k$ respectively.

In the case of the regular cones, using the method of upper and lower solutions and monotone iterative technique, S.Qi[4] has recently obtained the existence of maximal and minimal solutions of problem (1) when $\beta = 1$. As is known to all, the regularity of the cone is a very strict condition, and it is difficult to satisfy in applications. Later, using the Mönch fixed point theorem, X.Zhang[6] has studied the situation of problem(1) when $\beta > 1$, and has obtained the existence of solutions when the nonlinear term f and the impulsive function I_k satisfy the growth condition

$$\| f(t, u) \| \leq a(t) \| u \| + b(t),\ \forall\, t \in J,\ u \in E; \quad \| I_k(u) \| \leq C_k \| u \|, \tag{2}$$

where $a(t), b(t) \in C(J,\ \mathbb{R}^+) \cap L(J,\ \mathbb{R}^+),\ \mathbb{R}^+ = [0, +\infty),\ C_k$ are non-negative constants, $\sum\limits_{k=1}^{\infty} C_k$ is convergent, and satisfy

$$\frac{\beta}{\beta - 1}\Big[\int_0^\infty a(t)dt + \sum_{k=1}^{\infty} C_k\Big] < 1, \tag{3}$$

and noncompactness conditions

$$\alpha(f(t, D)) \leq l_1(t)\alpha(D),\ \forall\, t \in J, \tag{4}$$

$$\alpha(I_k(D)) \leq M_k\alpha(D),\ k = 1, 2, \cdots, \tag{5}$$

where $l_1(t) \in L(J,\ \mathbb{R}^+),\ M_k$ are non-negative constants, D is bounded set in E, and satisfy

$$2\frac{\beta}{\beta - 1}\int_0^\infty l_1(t)dt + \frac{\beta}{\beta - 1}\sum_{k=1}^{\infty} M_k < 1. \tag{6}$$

Inequality (3) and (6) are very strict restrictions, and they are difficult to satisfy in applications. The purpose of this paper is to extend and improve the results mentioned above. We will remove the non-compactness conditions (5) of the impulsive function I_k, and restrict conditions (3) and (6), and studies problem (1) only in the case of a regular cone.

2. Preliminaries

Let E be an ordered Banach space with the norm $\| \cdot \|$ and partial order$\leq$, whose posive cone P is normal with normal constant N. Let $J = [0, +\infty),\ 0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots,\ J_k = [t_{k-1}, t_k],\ k = 1, 2, \cdots,\ J' = J \backslash \{t_1, t_2, \cdots, t_k, \cdots\}$. $PC(J, E) = \{u : J \to E \mid u(t)$ is continuous at $t \neq t_k,$ and left continuous at $t = t_k,$ and $u(t_k^+)$ exists, $k = 1, 2, \cdots\},\ BPC(J, E) = \{u \in PC(J, E) : \sup_{t \in J} \| u(t) \| < \infty\}$. Evidently, $BPC(J, E)$ is a Banach space with norm $\| u \|_B = \sup_{t \in J} \| u(t) \|$. An abstract function $u \in BPC(J, E) \cap C^1(J', E)$ is called a solution of IVP(1) if $u(t)$ satisfies all the equalities of (1). Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of bounded set. For details of the definition and properties of the measure of noncompactness, please see [3].

Now, we first give the following lemmas in order to prove our main results.

Lemma1[3]. Let $B \subset C(I, E)$ be bounded and equicontinuous. Then $\alpha(B(t))$ is continuous on $I = [a, b]$, and

$$\alpha(B) = \max_{t \in I} \alpha(B(t)) = \alpha(B(I)).$$

Lemma2[2]. Let $B = \{u_n\} \subset BPC(I, E)$ be a bounded and countable set. Then $\alpha(B(t))$ is Lebesgue integral on $I,$ and

$$\alpha\Big(\Big\{ \int_I u_n(t) dt \mid n \in \mathbb{N} \Big\}\Big) \leq 2 \int_I \alpha(B(t)) dt.$$

Firstlywe consider the infinite boundary value problems of linear impulsive differential equation

$$\begin{cases} u'(t) + M(t)u(t) = h(t), & t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = y_k, \quad k = 1, 2, \cdots, \\ u(\infty) = \beta u(0), \end{cases} \tag{7}$$

where $h(t) \in BPC(J, E),\ M(t) \in C(J, \mathbb{R}^+) \cap L(J, \mathbb{R}^+)$.

Lemma3. For any $h(t) \in BPC(J, E),\ M(t) \in C(J, \mathbb{R}^+) \cap L(J, \mathbb{R}^+),\ M(t) \not\equiv 0,\ t \in J,\ x \in E$ and $y_k \in E,\ k = 1, 2, \cdots,$ the linear impulsive infinite boundary value problem (7) has a unique solution $u \in BPC(J, E) \cap$

$C^1(J', E)$ given by

$$u(t) = e^{-\int_0^t M(s)ds}\{\frac{1}{\beta e^{\int_0^\infty M(s)ds} - 1}[\int_0^\infty e^{\int_0^s M(\tau)d\tau}h(s)ds + \sum_{k=1}^{\infty} e^{\int_0^{t_k} M(s)ds}y_k]$$

$$+\int_0^t e^{\int_0^s M(\tau)d\tau}h(s)ds+\sum_{t_k<t} e^{\int_0^{t_k} M(s)ds}y_k\}. \tag{8}$$

Proof. For any $h(t) \in BPC(J, E)$, $M(t) \in C(J, \mathbb{R}^+) \cap L(J, \mathbb{R}^+)$, $M(t) \not\equiv 0$, $t \in J$, $x \in E$ and $y_k \in E$, $k = 1, 2, \cdots$. From the reference [8], it is easy to see the initial value problem of linear impulsive differential equation

$$\begin{cases} u'(t) + M(t)u(t) = h(t), \quad t \in J', \\ \Delta u|_{t=t_k} = y_k, \quad k = 1, 2, \cdots, \\ u(0) = x, \end{cases} \tag{9}$$

exists a unique solution $u \in BPC(J, E) \cap C^1(J', E)$ given by

$$u(t) = e^{-\int_0^t M(s)ds}[x+\int_0^t e^{\int_0^s M(\tau)d\tau}h(s)ds+\sum_{t_k<t} e^{\int_0^{t_k} M(s)ds}y_k]. \tag{10}$$

If u is a solution of the linear initial value problem (9) satisfy $u(\infty) = \beta x$, namely

$$\beta x = e^{-\int_0^\infty M(s)ds}[x + \int_0^\infty e^{\int_0^s M(\tau)d\tau}h(s)ds + \sum_{k=1}^{\infty} e^{\int_0^{t_k} M(s)ds}y_k],$$

then it is the solution of the linear impulsive infinite boundary value problem (7). Hence, we have

$$x = \frac{1}{\beta e^{\int_0^\infty M(s)ds} - 1}[\int_0^\infty e^{\int_0^s M(\tau)d\tau}h(s)ds + \sum_{k=1}^{\infty} e^{\int_0^{t_k} M(s)ds}y_k].$$

So, (8) is satisfied.

Inversely, we can verify directly that the function $u(t)$ defined by (8) is a solution of the linear impulsive infinite boundary value problem (7). □

Now, we define an operator A in $BPC(J, E)$ as following

$$\begin{aligned}(Au)(t) \;&=\; e^{-\int_0^t M(s)ds}[S(u) + \int_0^t e^{\int_0^s M(\tau)d\tau}(f(s, u(s)) + M(s)u(s))ds \\ &\quad + \sum_{t_k<t} e^{\int_0^{t_k} M(s)ds}I_k(u(t_k))],\end{aligned}$$

where

$$S(u) = \frac{1}{\beta e^{\int_0^\infty M(s)ds} - 1}[\int_0^\infty e^{\int_0^s M(\tau)d\tau}(f(s,u(s)) + M(s)u(s))ds + \sum_{k=1}^{\infty} e^{\int_0^{t_k} M(s)ds} I_k(u(t_k))].$$

Lemma4. Suppose that the condition

$(H1)$ There exist $p(t), q(t) \in C(J,\ \mathbb{R}^+) \cap L(J,\ \mathbb{R}^+)$ and the non-negative constants $C_k(k = 1, 2, \cdots)$, $\sum_{k=1}^{\infty} C_k$ is convergent, such that

$$\| f(t,u) \| \leq p(t) \| u \| + q(t), \forall\, t \in J, u \in E; \| I_k(u) \| \leq C_k \| u \|,\ k = 1, 2, \cdots.$$

is satisfied, then $A : BPC(J,E) \to BPC(J,E)$ and $u \in BPC(J,E) \cap C^1(J', E)$ is the solution of the BVP(1) if and only if $u \in BPC(J,E)$ is the fixed point of operator A.

Proof. Similar to the proof of lemma 4 in [4], $A : BPC(J,E) \to BPC(J,E)$, therefore, we easily know the operator A is defined. By the Lemma 3, it is easy to see that $u \in BPC(J,E) \cap C^1(J', E)$ is the solution of the BVP(1) if and only if $u \in BPC(J,E)$ is the fixed point of operator A. □

Definition1. If a function $v \in BPC(J,E) \cap C^1(J', E)$ satisfies that

$$\begin{cases} v'(t) \leq f(t, v(t)), \quad t \in J', \\ \Delta v|_{t=t_k} \leq I_k(v(t_k)), \quad k = 1, 2, \cdots, \\ v(\infty) \geq \beta v(0), \end{cases} \tag{11}$$

we call $v(t)$ a lower solution of the BVP(1); if all the inequalities of (11) are inverse, we call it an upper solution of the BVP(1).

For $\forall\ v, w \in BPC(J,E)$ with $v \leq w$, we use $[v, w]$ to denote the order interval $\{u \in BPC(J,E) \mid v \leq u \leq w\}$ in $BPC(J,E)$, and $[v(t), w(t)]$ to denote the order interval $\{u \in E \mid v(t) \leq u(t) \leq w(t),\ t \in J\}$ in E.

3. Main results

Theorem1. Let E be an ordered Banach space, whose positive cone P is normal, $f \in C(J \times E,\ E)$, $I_k \in C(E, E)$, $k = 1, 2, \cdots$. If the BVP(1) has a lower solution $v_0 \in BPC(J,E) \cap C^1(J', E)$ and an upper solution $w_0 \in BPC(J,E) \cap C^1(J', E)$ with $v_0 \leq w_0$, condition $(H1)$ hold, and satisfy:

$(H2)$ There exists $M(t) \in C(J, \mathbb{R}^+) \cap L(J, \mathbb{R}^+)$, $M(t) \not\equiv 0$, such that

$$f(t, u_2) - f(t, u_1) \geq -M(t)(u_2 - u_1); \quad I_k(u_2) \geq I_k(u_1),\ k = 1, 2 \cdots;$$

for $\forall\ t \in J,$ and $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$.

$(H3)$ There exists $l(t) \in C(J, \mathbb{R}^+) \cap L(J, \mathbb{R}^+),$ such that

$$\alpha(\{f(t, u_n)\}) \leq l(t)\alpha(\{u_n\});$$

for $t \in J$and increasing or decreasing monotonic sequences$\{u_n\} \subset [v_0(t), w_0(t)]$.

$(H4)$ The sequences $v_n(0)$ and $w_n(0)$ are convergent, where $v_n = Av_{n-1},$ $w_n = Aw_{n-1},\ n = 1, 2, \cdots$.

Then the BVP(1) has minimal solution $\underline{u}$ and maximal solution $\overline{u}$ in $[v_0, w_0]$, which can be obtained by a monotonic iterative procedure starting from v_0 and w_0 respectively.

Proof. By the definition of operator $A,$ the assumptions $(H1)$ and $(H2)$, we know that $A : [v_0, w_0] \to BPC(J, E)$ is continuous, and A is a monotone increasing operator.

We show that $v_0 \leq Av_0,\ Aw_0 \leq w_0$.Let $h(t) = v_0'(t) + M(t)v_0(t)$, by (11), $h \in BPC(J, E) \cap C^1(J', E)$ and $h(t) \leq f(t, v_0(t)) + M(t)v_0(t),\ t \in J'$. Hence, by the Lemma 3

$$\begin{aligned} v_0(t) &= e^{-\int_0^t M(s)ds}\Big\{\frac{1}{\beta e^{\int_0^\infty M(s)ds} - 1}\Big[\int_0^\infty e^{\int_0^s M(\tau)d\tau}h(s)ds + \sum_{k=1}^{\infty} e^{\int_0^{t_k} M(s)ds} \\ &\quad \Delta v_0|_{t=t_k}\Big] + \int_0^t e^{\int_0^s M(\tau)d\tau}h(s)ds + \sum_{t_k<t} e^{\int_0^{t_k} M(s)ds}\Delta v_0|_{t=t_k}\Big\} \\ &\leq e^{-\int_0^t M(s)ds}\Big[S(v_0) + \int_0^t e^{\int_0^s M(\tau)d\tau}(f(s, v_0(s)) + M(s)v_0(s))ds \\ &\quad + \sum_{t_k<t} e^{\int_0^{t_k} M(s)ds}I_k(v_0(t_k))\Big] \\ &= Av_0(t), \quad t \in J. \end{aligned}$$

namely, $v_0 \leq Av_0$. Similarly, it can be show that $Aw_0 \leq w_0$. So, $A : [v_0, w_0] \to [v_0, w_0]$ is a continuously increasing operator.

Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v_0, w_0]$ by the iterative schem

$$v_n = Av_{n-1}, \quad w_n = Aw_{n-1}, \quad n = 1, 2, \cdots \tag{12}$$

Then from the monotonicity of $A,$ it follows that

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_2 \leq w_1 \leq w_0. \tag{13}$$

We prove that $\{v_n\}$ and $\{w_n\}$ are uniformly convergent in J.

For convenience, let $B = \{v_n \mid n \in \mathbb{N}\},\ \ B_0 = \{v_{n-1} \mid n \in \mathbb{N}\}$. Since $B = A(B_0),$ by the definition of the operator A and the boundedness of $B_0,$ we easily see that B is equicontinuous in every interval $J_k',$ where $J_1' = [0, t_1],\ J_k' =$

$(t_{k-1}, t_k],\ k = 2, 3, \cdots$. From $B_0 = B \cup \{v_0\}$ it follows that $\alpha(B_0(t)) = \alpha(B(t)),\ t \in J$. Let $\varphi(t) = \alpha(B(t)),\ t \in J,$ by the Lemma 1, $\varphi \in BPC(J, \mathbb{R}^+)$. Next, we show that $\varphi(t) \equiv 0$ in every interval $J'_k\ (k = 1, 2 \cdots)$.

For $t \in J'_1$, by the definition of the operator $A,$ assumptions $(H3),\ (H4)$ and the Lemma 2, we have

$$\begin{aligned}
\varphi(t) &= \alpha(B(t)) = \alpha(A(B_0)(t)) = \alpha\Big(\Big\{e^{-\int_0^t M(s)ds} S(v_{n-1}) \\
&\quad + \int_0^t e^{-\int_s^t M(\tau)d\tau}[f(s, v_{n-1}(s)) + M(s)v_{n-1}(s)]ds\Big\}\Big) \\
&\le \alpha(\{e^{-\int_0^t M(s)ds} v_n(0)\}) \\
&\quad +2\int_0^t e^{-\int_s^t M(\tau)d\tau} \alpha(\{f(s, v_{n-1}(s)) + M(s)v_{n-1}(s)\})ds \\
&\le 2\int_0^t (l(s) + M(s))\varphi(s)ds.
\end{aligned}$$

Hence by the Belman inequality, $\varphi(t) \equiv 0$ in J'_1. In particular, $\alpha(B(t_1)) = \alpha(B_0(t_1)) = 0,$ this means that $B(t_1)$ and $B_0(t_1)$ are precompact in E. Thus I_1 $(B_0(t_1))$ is precompact in E, and $\alpha(I_1(B_0(t_1))) = 0$.

For $t \in J'_2,$ by the definition of the operator A and the above argument for $t \in J'_1,$ we have

$$\begin{aligned}
\varphi(t) &= \alpha(B(t)) = \alpha(A(B_0)(t)) \\
&\le \alpha\Big(\Big\{e^{-\int_0^t M(s)ds} S(v_{n-1}) + \int_0^t e^{-\int_s^t M(\tau)d\tau}[f(s, v_{n-1}(s)) \\
&\quad +M(s)v_{n-1}(s)]ds\Big\}\Big) + \alpha(\{I_1(v_{n-1}(t_1))\}) \\
&\le 2\int_0^t (l(s) + M(s))\varphi(s)ds + \alpha(I_1(B_0(t_1))) \\
&= 2\int_{t_1}^t (l(s) + M(s))\varphi(s)ds.
\end{aligned}$$

Again by the Belman inequality, $\varphi(t) \equiv 0$ in $J'_2,$ from which we obtain that $\alpha(B_0(t_2)) = 0$ and
$\alpha(I_2(B_0(t_2))) = 0$.

Suppose $t \in J'_k$ such that $\varphi(t) \equiv 0$ and $\alpha(B_0(t_k)) = 0,\ \alpha(I_k(B_0(t_k))) = 0$. For $t \in J'_{k+1},$ we have

$$\begin{aligned}\varphi(t) &= \alpha(B(t)) = \alpha(A(B_0)(t))\\ &\leq \alpha\Big(\Big\{e^{-\int_0^t M(s)ds}S(v_{n-1}) + \int_0^t e^{-\int_s^t M(\tau)d\tau}[f(s, v_{n-1}(s))\\ &\quad +M(s)v_{n-1}(s)]ds\Big\}\Big) + \alpha(\{I_k(v_{n-1}(t_k))\})\\ &\leq 2\int_0^t (l(s) + M(s))\varphi(s)ds + \alpha(I_k(B_0(t_k)))\\ &= 2\int_{t_k}^t (l(s) + M(s))\varphi(s)ds.\end{aligned}$$

Hence by the Belman inequality $\varphi(t) \equiv 0$ in J'_{k+1} and $\alpha(B_0(t_{k+1})) = 0,\ \alpha(I_{k+1}(B_0(t_{k+1}))) = 0$.

So, by the mathematical induction, we can prove that $\varphi(t) \equiv 0$ in every J'_k $(k = 1, 2 \cdots)$.

For $J_k,$ if we modify the value of v_n at $t = t_{k-1}$ via $v_n(t_{k-1}) = v_n(t_{k-1}^+),\ n \in \mathbb{N},$ then $\{v_n\} \subset C(J_k, E)$ and it is equicontinuous. Since$\alpha(\{v_n(t)\}) = 0, \{v_n(t)\}$ is precompact in E for every $t \in J_k$. By the Arzela-Ascoli theorem, $\{v_n\}$ is precompact in $C(J_k, E)$. Hence, $\{v_n\}$ has a convergent subsequence in $C(J_k, E)$. Combing this with the monotonicity (13), we easily prove that $\{v_n\}$ itself is convergent in $C(J_k, E)$. In particular, $\{v_n(t)\}$ is uniformly convergent in J'_k. Consequently, $\{v_n(t)\}$ is uniformly convergent over the whole of J. Similarly, we can prove that $\{w_n(t)\}$ is also uniformly convergent in J. Hence, $\{v_n\}$ and$\{w_n\}$ are convergent in $BPC(J, E)$. Set

$$\underline{u} = \lim_{n\to\infty} v_n, \qquad \overline{u} = \lim_{n\to\infty} w_n, \qquad \text{in } BPC(J, E).$$

Letting $n \to \infty$ in (12) and (13), we see that $v_0 \leq \underline{u} \leq \overline{u} \leq w_0$ and $\underline{u} = A\underline{u}$, $\overline{u} = A\overline{u}$. Combining this with the monotonicity of operator A, it is easy to see that $\underline{u}$ and $\overline{u}$ are the minimal and maximal fixed points of A in $[v_0, w_0]$, and therefore, they are the minimal and maximal solutions of BVP(1) in $[v_0, w_0]$, respectively. □

In Theorem 1, if E is weakly sequentially complete, the conditions $(H3)$ and $(H4)$ hold automatically. In fact, by [7, Theorem 2.2], any monotonic and order-bounded sequence is precompact. By the monotonicity (13) and the same method in proof of Theorem 1, we can easily see that $\{v_n(t)\}$ and $\{w_n(t)\}$ are convergent on J. In particular, $\{v_n(0)\}$ and $\{w_n(0)\}$ are convergent. So, condition $(H4)$ holds. Let $\{u_n\}$ be a increasing or decreasing sequence obeying condition $(H3)$, then by condition $(H2)$, $\{f(t, u_n) + M(t)u_n\}$ is a monotonic and order-bounded sequence. By the property of measure of noncompactness, we have

$$\alpha(\{f(t, u_n)\}) \leq \alpha(\{f(t, u_n) + M(t)u_n\}) + M(t)\alpha(\{u_n\}) = 0.$$

Hence, condition $(H3)$ holds. From the Theorem 1, we obtain

Corollary1. Let E be an ordered and weakly sequentially complete Banach space,
whose positive cone P is normal, $f \in C(J \times E,\ E)$, $I_k \in C(E,E)$, $k = 1,2,\cdots$. If the BVP(1) has a lower solution $v_0 \in BPC(J,E) \cap C^1(J',E)$ and an upper solution $w_0 \in BPC(J,E) \cap C^1(J',E)$ with $v_0 \le w_0$, and conditions $(H1)$ and $(H2)$ are satisfied, then the BVP(1) has minimal and maximal solutions in $[v_0, w_0]$, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively.

Remark1. By the reference [7], we know that the cone in weakly sequentially complete Banach space is normal. So, the Corollary 1 is the main result in [4].

If we replace the assumption $(H3)$ by the following assumption:

$(H5)$ There exist $\overline{M}(t) \in C(J,\mathbb{R}^+) \cap L(J,\mathbb{R}^+)$, such that

$$f(t,u_2) - f(t,u_1) \le \overline{M}(t)(u_2 - u_1),$$

$$\forall\ t \in J, \text{ and } v_0(t) \le u_1 \le u_2 \le w_0(t),$$

we have the following result:

Theorem2. Let E be an ordered Banach space, whose positive cone P is normal, $f \in C(J \times E,\ E)$, $I_k \in C(E,E)$, $k = 1,2,\cdots$. If the BVP(1) has a lower solution $v_0 \in BPC(J,E) \cap C^1(J',E)$ and an upper solution $w_0 \in BPC(J,E) \cap C^1(J',E)$ with $v_0 \le w_0$, such that conditions $(H1)$, $(H2)$, $(H4)$ and $(H5)$ hold, then the BVP(1) has minimal solution $\underline{u}$ and maximal solution $\overline{u}$ in $[v_0, w_0]$, which can be obtained by a monotonic iterative procedure starting from v_0 and w_0 respectively.

Proof. For $t \in J$, let $\{u_n\} \subset [v_0(t), w_0(t)]$ be a increasing sequence. For $m, n \in \mathbb{N}$ with $m > n$, by $(H2)$ and $(H5)$, we have

$$\theta \le f(t,u_m) - f(t,u_n) + M(t)(u_m - u_n) \le (\overline{M}(t) + M(t))(u_m - u_n).$$

By this and the normality of cone P, we have

$$\| f(t,u_m) - f(t,u_n) \| \le (N(\overline{M}(t) + M(t)) + M(t)) \| u_m - u_n \| .$$

From this inequality and the definition of the measure of noncompactness, it follows that

$$\alpha(\{f(t,u_n)\}) \le l(t)\alpha(\{u_n\}),$$

where $l(t) = N(\overline{M}(t) + M(t)) + M(t)$. If $\{u_n\}$ is a decreasing sequence, the above inequality is also valid. Hence the condition $(H3)$ hold.

Therefore, by the Theorem 1, the BVP(1) has minimal solution $\underline{u}$ and maximal solution $\overline{u}$ in $[v_0, w_0]$, which can be obtained by a monotonic iterative procedure starting from v_0 and w_0 respectively. $\square$

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