Int. J. Contemp. Math. Sciences, Vol. 6, 2011, no. 24, 1151 - 1158

Commuting Derivations of Semiprime Rings

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Abstract

The main purpose of this paper is to study and investigate some results concerning generalized derivation D on semiprime ring R, we obtain a derivation d is commuting on R.

Mathematics Subject Classification: 16W25, 16N60, 16U80.

Keywords: Semiprime rings,derivations, generalized derivation, commuting derivation.

This research has been motivated by the work of M.Ashraf[1]and M.A.Quadri,M.Shadab Khan and N.Rehman[13].Throughout this paper,R will represent an associative ring and has a cancellation property with the center Z(R).We recall that R is semiprime if $xRx = (o)$ implies $x=o$ and it is prime if $xRy=(o)$ implies $x=o$ or $y=o$.A prime ring is semiprime but the converse is not true in general . A ring R is 2-torsion free in case $2x = o$ implies that $x = o$ *for any* $x \in R$ .An additive mapping $d:R \to R$ is called a derivation if $d(xy)=d(x)y+xd(y)$ *holds for all* $x,y \in R$.A mapping d is called centralizing if $[d(x),x] \in Z(R)$ for all $x \in R$, in particular, if $[d(x),x] = o$ for all $x \in R$, then it is called commuting,and is called central if $d(x) \in Z(R)$ *for all* $x \in R$.Every central mapping is obviously commuting but not conversely in general.In[4]Q.Deng and H.E.Bell extended the notion of commutativity to one of n-commutativity,where n is an arbitrary positive integer,by defining a mapping d to be n-commuting on U if $[x^n,d(x)]=0$ *for all* $x \in U$,where U be a non empty subset of R .Following Bresar [3] an additive mapping $D:R \to R$ is called a generalized derivation on R if there exists a derivation d:R→R such that $D(xy)=D(x)y+xd(y)$ *holds for all* $x,y \in R$.However , generalized derivation covers the concept of derivation.Also with d=o,a generalized derivation covers the concept of left multiplier (left centralizer) that is,an additive mapping D satisfying $D(xy) = D(x)y$ *for all* $x,y \in R$.As usual,we write $[x,y]$ *for* $xy - yx$ and make use of the commutator identities $[xy,z]=x[y,z]+[x,z]y$ *and* $[x,yz]=y[x,z]+ [x,y]z$,and the symbol *xoy* stands for the anti-commutator $xy+yx$. Generalized derivation of operators on various algebraic structures have been an active area of research since the last fifty years due to their usefulness in various fields of mathematics.Some authors have studied centralizers in the general framework of semiprime rings (see[2,13,14,15,16and19]).Muhammad A.C.and Mohammed S.S.[10] proved,let R be a semiprime ring and $d:R \to R$ a mapping satisfy $d(x)y = xd(y)$ *for all* $x,y \in R$.Then d is a centralizer.Molnar [9] has proved,let R be a 2-torsion free prime ring and let d:R → R be an additive mapping.If $d(xyx)=d(x)yx$ *holds for every* $x,y \in R$,then d is a left centralizer.Muhammad A.C. and A. B.Thaheem [11] proved ,let d and g be a pair of derivations of semiprime ring R satisfying $d(x)x+xg(x) \in Z(R)$,then cd and cg are central for all $c \in Z(R)$.J.Vukman [18] proved,let R be a 2-torsion free semiprime ring and let d:R →R be an additive centralizing mapping on R, in this case,d is commuting on R .B.Zalar [19] has proved, let R be a 2-torsion free semiprime ring and d:R → R an dditive mapping which satisfies $d(x^2)= d(x)x$ for all $x \in R$.Then d is a left centralizer.Hvala [5] initiated the algebraic study of generalized derivation and extended some results concerning derivation to generalized derivation . Majeed and Mehsin [6] proved,let R be a 2- torsion free semiprime ring, (D,d) and (G,g) be generalized derivations of R,if R admits to satisfy $[D(x),G(x)]= [d(x),g(x)]$ *for all* $x \in R$ and d acts as a left centralizer, then (D,d) and (G,g) are orthogonal generalized derivations of R.Recently,Mehsin Jabel [7] proved, let R be a semiprime ring and U be a non-zero ideal of R.If R admits a generalized derivation D associated with a non-zero derivation d such that $D(xy) - yx \in Z(R)$ *for all* $x,y \in U$,then R contains a non-zero central ideal.Mehsin Jabel[8] proved, let R be a semiprime ring with left cancellation proprety,(D,d) and (G,g) be a non-zero generalized derivations of R,U a non-zero ideal of R,if R admists to satisfy $[d(x),g(x)]=o$ *for all* $x \in U$ and a non-zero d acts as a left centralizer (resp.a non-zero g acts as a left centralizer),then R contains a non-zero central ideal,where two a dditive maps $d,g:R\to R$ are called orthogonal if $d(x)Rg(y) =o=g(y)Rd(x)$ *for all* $x,y \in R$.And two generalized derivations (D,d) and (G,g) of R are called orthogonal if $D(x)RG(y)=o=G(y)RD(x)$ *for all* $x,y \in R$,and we denote by (D,d) to a generalized derivation $D:R\to R$ determined by a derivation d of R .In this paper we study and investigate some results concerning generalized derivation D on semiprime ring R,we give some results about that.

Lemma 1 [12,Corollary 9]
Any anticommutative semiprime ring R is commutative,where A ring R is said to be anticommutative if $xy=-yx$ (that is, $xy + yx = 0$) for all $x,y \in R$.

Theorem 2.1

Let R be a semiprime ring.If R admits a non-zero generalized derivation D associated with a non-zero derivation d such that $D([x,y])=[x,y]$ for all $x,y \in R$.Then d is commuting on R.

Proof: For any $x,y \in R$,we have $D([x,y])=[x,y]$ for all $x,y \in R$,which gives

$D(x)y+xd(y)-D(y)x-yd(x)-[x,y]=0$ for all $x,y \in R$ (2.1)

Replacing y by yz in (2.1),we obtain

$D(x)yz+xd(y)z+xyd(z)-D(y)zx-yd(z)x-yzd(x)-y[x,z]-[x,y]z=0$ for all $x,y \in R$. (2.2)

Substituting (2.1) in (2.2) gives

$D(y)[x,z]+yd(x)z +xd(y)z+xyd(z)-yd(z)x-yzd(x)-y[x,z]=0$ for all $x,y \in R$. (2.3)

Replacing z by x in (2.3),we obtain

$xd(y)x+xyd(x)-yxd(x)=0$ for all $x,y \in R$. (2.4)

Replacing y by x in (2.4),we get

$xd(x)x=0$ for all $x \in R$. (2.5)

By using the cancellation property on x ,from left,we obtain

$d(x)x=0$ for all $x \in R$. (2.6)

Again by using the cancellation property on x ,from right,we get

$xd(x)=0$ for all $x \in R$. (2.7)

Subtracting (2.6) and (2.7) ,we obtain

$[d(x),x]=o$ for all $x \in R$.Thus ,d is commuting on R,by this we complete our proof.

A slight modification in the proof of the above theorem yields the following .

Theorem 2.2
Let R be a semiprime ring.If R admits a non-zero generalized derivation D associated with a non-zero derivation d such that $D([x,y])+[x,y]=0$ for all $x,y \in R$.Then d is commuting on R.

Theorem 2.3

Let R be a semiprime ring.If R admits a non-zero generalized derivation D associated with a non-zero derivation d such that $D(xoy)=(xoy)$ for all $x,y \in R$.Then d is 2-commuting on R.

Proof: For any $x,y \in R$,we have

$D(xoy)=(xoy)$ for all $x,y \in R$.

This can be written as

$D(x)y+xd(y)+D(y)x+yd(x)-(xoy)=0$ for all $x,y \in R$. (2.8)

Replacing y by yx in above equation ,we obtain

$D(x)yx+xd(y)x+xyd(x)+D(y)x^2+yd(x)x-(xoy)x=0$ for all $x,y \in R$. (2.9)

According to (2.8) the relation above reduced to

$(xoy)d(x)=0$ for all $x,y \in R$.

By using the cancellation property on d(x),we get

$(xoy)=0$ for all $x,y \in R$. (2.10)

By Lemma 1,we get

$[x,y]=0$ for all $x,y \in R$. (2.11)

Replacing x by x^2 and y by d(x),we get

$[d(x),x^2]=0$ for all $x \in R$.Thus,d is 2-commuting on R.

We complete our proof.

A slight modification in the proof of the Theorem 2.3 , yields the following

Theorem 2.4

Let R be a semiprime ring.If R admits a non-zero generalized derivation D associated with a non-zero derivation d such that $D(xoy)+(xoy)=0$ for all $x,y \in R$.Then d is 2-commuting on R .

Proposition 2.5

Let R be a semiprime ring.If R admits a non-zero ideal generalized derivation D associated with a non-zero derivation d such that $D([x,y])\pm(xoy)=0$ for all $x,y \in R$.Then 2- d is commuting on R .

Proof: For any $x,y \in R$, we have

$D([x,y])-(xoy)=0$.Then

$[D([x,y]),r]-[(xoy),r]=0$ for all $x,y,r \in R$.

Replacing y by x,we obtain

$2[x^2,r]=0$ for all $x,r \in R$.By using the cancellation property with replacing r by d(x) ,we obtain ,2-d is commuting on R.

Theorem 2.6

Let R be a semiprime ring.If R admits a non-zero generalized derivation D associated with a non-zero derivation d such that d(x)oD(y)=0 for all x,y $\in$ R.Then d is commuting on R .

Proof: We have $d(x)oD(y)=0$ for all $x,y \in R$. (2.12)

Replacing y by yr,we obtain

$(d(x)oy)d(r)-y[d(x),d(r)]+(d(x)oD(y))r-D(y)[d(x),r]=0$ for all $x,y \in U, r \in R$. (2.13)

According to (2.12),then (2.13) reduced to

$(d(x)oy)d(r)-y[d(x),d(r)]-D(y)[d(x),r]=0$ for all $x,y \in U, r \in R$.
Replacing r by d(x),we get

$$(d(x)oy)d^2(x)-y[d(x),d^2(x)]=0 \quad \text{for all } x,y \in R. \tag{2.14}$$

Replacing y by zy in (2.14), with using (2.14),we obtain

$$[d(x),z]yd^2(x)=0 \text{ for all } x,y,z \in R. \tag{2.15}$$

By using the cancellation property on (2.15) ,from right,we obtain

$$[d(x),z]y=0 \text{ for all } x,y,z \in R. \tag{2.16}$$

Since R is semiprime from above relation ,we get

$$[d(x),z]=0 \text{ for all } x,y \in R. \tag{2.17}$$

Replacing z by x,we obtain, d is commuting on R.

Theorem 2.7

Let R be a semiprime ring.If R admits a non-zero generalized derivation D associated with a non-zero derivation d such that $[d(x),D(y)]=0$ for all $x,y \in R$.Then d is 2-commuting on R .

Proof: We have $[d(x),D(y)]=0$ for all $x,y \in R$. (2.18)
Replacing y by yz in(2.18)and using the result with (2.18),we obtain

$$D(y)[d(x),z]+y[d(x),d(z)]+[d(x),y]d(z)=0 \text{ for all } x,y \in R. \tag{2.19}$$

Replacing z by zd(x) in (2.19) and using the result with (2.19),we get

$$yz[d(x),d^2(x)]+y[d(x),z]d^2(x)+[d(x),y]zd^2(x)=0 \quad \text{for all } x,y \in R. \tag{2.20}$$

Again replacing y by ry in (2.20) and using the result with (2.20),we obtain

$[d(x),z]yd^2(x)=0$ for all $x,y,z \in R$.

By using similar arguments as in the proof of Theorem 2.6,we obtain the required result.

Theorem 2.8

Let R be a semiprime ring.If R admits a non-zero generalized derivation D associated with a non- zero derivation d such that d(x)oD(y)=xoy for all x,y $\in$ R.Then d is 2-commuting on R .

Proof: For any $x,y \in R$,we have

$d(x)oD(y)=xoy$ for all $x,y \in R$.Replacing y by yr,we get

$(d(x)oy)d(r)-y[d(x),d(r)]+(d(x)oD(y))r-D(y)[d(x),r]=(xoy)r-y[x,r]$ for all $x,y,r \in R$.

Using our relation ,we obtain

$$(d(x)oy)d(r)-y[d(x),d(r)]-D(y)[d(x),r]+y[x,r]=0 \text{ for all } x,y,r \in R. \tag{2.21}$$

In (2.21) replacing r by d(x),we obtain

$$(d(x)oy)d^2(x)-y[d(x),d^2(x)]+y[x,d(x)]=0 \quad \text{for all } x,y \in R. \tag{2.22}$$

Replacing y by zy in (22),we obtain

$$(z(d(x)oy)+[d(x),z]y)d^2(x)-zy[d(x),d^2(x)]+zy[x,d(x)]=0 \text{ for all } x,y \in R. \tag{2.23}$$

According to(2.22),above relation reduced to

$$[d(x),z]yd^2(x)=0 \quad \text{for all } x,y,z \in R. \tag{2.24}$$

By using similar arguments as in the proof of Theorem 2.6,we obtain the required result.

A slight modification in the proof of the Theorem2.8 , yields the following

Theorem 2.9

Let R be a semiprime ring.If R admits a non-zero generalized derivation D associated with a non- zero derivation d such that $d(x)oD(y)+xoy=0$ for all $x,y \in R$.Then d is 2-commuting on R .

Theorem 2.10

Let R be a semiprime ring.If R admits a non-zero generalized derivation D associated with a non-zero derivation d such that $d(x)D(y)-xy \in Z(R)$ for all $x,y \in R$.Then d is commuting on R.

Proof: For any $x,y \in R$,we have

$d(x) D(y)-xy \in Z(R)$,replacing y by yr,we obtain

$(d(x)D(y)-xy)r+d(x)yd(r) \in Z(R)$ for all $x,y ,r \in R$. (2.25)

This implies that

$[d(x)yd(r),r]=0$ for all $x,y ,r \in R$. (2.26)

Hence it follows that

$d(x)[yd(r),r]+[d(x),r]yd(r)=0$ for all $x,y ,r \in R$. (2.27)

In (2.27) replacing y by d(x)y,we obtain

$[d(x),r]d(x)yd(r)=0$ for all $x,y,r \in R$. (2.28)

By using the cancellation property on d(x)yd(r),we obtain

$[d(x),r]=0$ for all $x,r \in R$. (2.29)

Replacing r by x in above relation,we obtain

$[d(x),x] =0$ for all $x \in R$. (2.30)

Then according to (2.30),we obtain
d is commuting on R.
By same method in above theorem ,we can prove the following.

Theorem 2.11

Let R be a semiprime ring.If R admits a non-zero generalized derivation D associated with a non-zero derivation d such that $d(x)D(y)+xy \in Z(R)$ for all $x,y \in R$.Then d is commuting on R .

Theorem 2.12

Let R be a semiprime ring.If R admits a non-zero generalized derivation D associated with a non-zero derivation d such that $[d(x),D(y)]=[x,y]$ for all $x,y \in R$.Then d is 2-commuting on R .

Proof: For any $x,y \in R$,we have

$[d(x),D(y)]=[x,y]$ for all $x,y \in R$. (2.31)

Replacing y by yz in (2.31),with using the result with (2.31),we obtain

$D(y)[d(x),z]+y[d(x),d(z)]+[d(x),y]d(z)=y[x,z]$ for all $x,y \in R$. (2.32)

Again replacing z by zd(x) in (2.32) with using the result with (2.32),we obtain

$y[d(x),z]d^2(x)+yz[d(x),d^2(x)]+[d(x),y]zd^2(x)=yz[x,d(x)]$ for all $x,y \in R$. (2.33)

Replacing y by ry in(2.33),we obtain

$ryz[d(x),d^2(x)]+ry[d(x),z]d^2(x)+r[d(x),y]zd^2(x)+[d(x),r]yzd^2(x)=ryz[x,d(x)]$

for all $x,y,r \in R$. (2.34)

According to (2.33),the relation(2.34) reduced to

$[d(x),r]yzd^2(x)=0$ for all $x,y,r \in R$. (2.35)

Thus by same method in Theorem 2.6,we complete our proof.

Proceeding on the same lines with necessary variations,we can prove the following.

Theorem 2.13

Let R be a semiprime ring.If R admits a non-zero generalized derivation D associated with a non-zero derivation d such that $[d(x),D(y)]+[x,y]=0$ for all $x,y \in R$.Then d is 2-commuting on R .

Acknowledgements. The authors are greatly indebted to the referee for his careful reading the paper.

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Received: November, 2010