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Covering Graph for Diagram Groups

from Semigroup Presentation

$\mathbf{S} =< \mathbf{a}, \mathbf{b} : \mathbf{a} = \mathbf{b} >$

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Abstract

The aim of this paper is to determine all connected $2-$complex graphs Γ_i that obtained from semigroup presentation $S =< a, b : a = b >$. Then we prove that Γ_{i+1} is the covering space (covering graphs) for Γ_i for all $i \in N$.

Keywords: semigroup, semigroup presentation, mapping of $2-$complex, covering graph, diagram group

Introduction

In this section we explain briefly about semigroup presentation and diagram groups which are useful for our purpose. Let $S =< X : r >$ be a semigroup presentation, where X is a set of generator where elements of relations in r is of the form $R_\varepsilon = R_{-\varepsilon}$ ($R_{\pm\varepsilon}$are reduced positive words on X). We may construct the diagram group $D(S, W)$ where W is a positive word on X as described for example in Ahmad, (2003) and Guba and Sapir (1997) and Kilibarda (1997).
A $2-$complex is a graph $(V, E, i, \tau, -1)$ with a set r of $2-$cells and a mapping which maps each $2-$cells from r to a closed path on this graph which is called the defining path of this $2-$cell. For any semigroup presentation S, we may obtain a

$2-$complex K(S). Vertices of K(S) are all positive words on X^* while edges are atomic pictures labeled by $A = (u, m \rightarrow n, v)$ where u, v are words on X^* and $(m = n) \in r$. The 2-cell of K(S) are 5-tuples of the form $(u, m_1 \rightarrow n_1, v, m_2 \rightarrow n_2, w)$ where $u, v, w \in X^*$ and $(m_i = n_i) \in r$.
Such a 2-cell has the following defining specific path:

$$(um_1v, m_2 \rightarrow n_2, w)(u, m_1 \rightarrow n_1, vn_2w)(un_1v, m_2 \rightarrow n_2, w)^{-1}(u, m_1 \rightarrow n_1, vm_2w)^{-1}.$$

It is easy to see that 2-cell correspond to independent applications of the relations from r. See Guba and Sapir (1997) for details. Diagram groups are considered from geometrical objects called semigroup diagrams. These diagrams are drawn and considered as 2-complex graphs. A particular group can be developed from a given graph. This group is called a diagram group.

This paper determines all the connected 2-complex graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ from semigroup presentation $S = < a, b : a = b >$. We prove that Γ_{i+1} is the covering graph of Γ_i, for all $i \in N$.

In next section we explain briefly about words, graphs, semigroup presentations, atomic pictures, pictures and diagram groups which are useful for our purpose.

2 Basic Definitions

Definition 2.1 Let $S = < X : r >$ be a semigroup presentation a word W on X is defined to be of the form $x_1^{\varepsilon_1}x_2^{\varepsilon_2}\ldots x_n^{\varepsilon_n}$ such that $n \geq 0, x_i \in X, \varepsilon = 1$. Two positive words are equivalent if one can be obtained from the other using a finite number of elementary operations. The equivalent class containing the word W will be denoted by [W]. The product of two words U and V is defined by writing the word U and then followed by the word V, we denote this product as U V.

Theorem 2.2 The algebraic system $\{[\alpha] : \gamma \text{ is a path}\}$ with binary operation $[\alpha].[\beta] = [\alpha\beta]$. The identity of this semigroup is [1], while the inverse $[\alpha]^{-1} = [\alpha^{-1}]$. This semigroup is known as the free group on X. [2]

Definition2.3 A graph Γ consists of five pair $(V, E, i, \tau, -1)$ where V and E are two disjoint finite sets. Set V is known as the set of vertices while E as the set of edges. Symbols $i, \tau, -1$ are functions $i : E \rightarrow V, \tau : E \rightarrow V, -1 : V \rightarrow V$ such that $i(e) = \tau(e^{-1}), \tau(e) = i(e^{-1}), e \neq e^{-1}, e \in E$. The function i and τ are known as the initial and the terminal functions respectively. A graph Γ is connected if given any two vertices in Γ, there is a path joining them. A path γ in the graph Γ is of the form $e_1^{\varepsilon_1}e_2^{\varepsilon_2}..e_n^{\varepsilon_n}$, $n \geq 0, e_i \in E, \varepsilon = \pm 1$ such that $\tau(e_i^{\varepsilon_i}) = i(e_{i+1}^{\varepsilon_{i+1}})$. The path γ is closed if $i(\gamma) = \tau(\gamma)$. Let γ and β be two paths in the graph Γ. If $\tau(\gamma) = i(\beta)$ then the product of γ with β is defined by tracing of γ then followed by β, denoted by γβ. Two paths γ and β are equivalent if γ can obtained from β by using a finite number of elementary opertations.

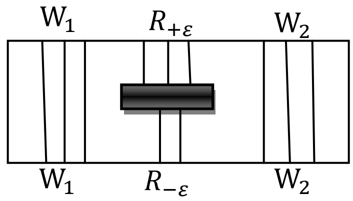
Definition2.4 A $2-$complex K is a pair $<\Gamma : r>$ where Γ is a graph and r is a set of closed paths in Γ. This $2-$complex is finite if Γ is finite and is connected If Γ is connected. The equivalent class containing the path γ will be denoted by $[\gamma]$.

Theorem2.5 Let K be a connected $2-$complex and fix a vertex v. The algebraic system $\pi_1(K, v) = \{ [\gamma] : i(\gamma) = \tau(\gamma) = v \}$ with binary operation $[\gamma] \cdot [\beta] = [\gamma\beta]$ forms a group called the first fundamental group with base point v where γ, β are closed paths in Γ. Since the fundamental group of a connected $2-$complex graph is independent of chosen vertex, we simply write $\pi_1(K)$. [2]

Definition2.6 Let $S = < X : r >$ be a semigroup presentation. An atomic picture A over S is of the form

$A = (W_1, R_{+\varepsilon} \to R_{-\varepsilon}, W_2)$

$W_i \in X^*$, $R_\varepsilon = R_{-\varepsilon} \in r$

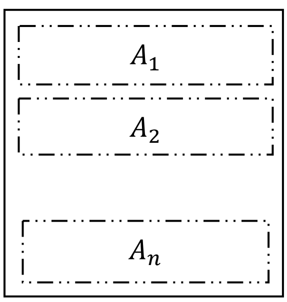


$\leftarrow i(A) = W_1 R_{+\varepsilon} W_2$

$\leftarrow \tau(A) = W_1 R_{-\varepsilon} W_2$

Figure 1 Atomic picture over S

Definition2.7 A picture P over a semigroup presentation S is a collection of atomic pictures $A_1, A_2, \dots, A_n$ such that $\tau(A_i) = i(A_{i+1})$, $i = 1, \dots, n-1$.



$\leftarrow i(P) = i(A_1)$

$\leftarrow \tau(P) = \tau(A_n)$

Figure 2 A picture over S

Definition2.8 Let $\Gamma_1 = (V_1, E_1, i, \tau, -1)$ and $\Gamma_2 = (V_2, E_2, i, \tau, -1)$ be 2 −complex graphs. A mapping $\Omega : \Gamma_1 \to \Gamma_2$ is a function from $V_1 \cup E_1 \to V_2 \cup E_2$ sending vertices to vertices such that $\Omega(V_1) \subseteq V_2$, edges to edges such that $\Omega(E_1) \subseteq E_2$ and respecting incidences and inversions

$$\Omega(i(e)) = i(\Omega(e)) \ , \quad \Omega(\tau(e)) = \tau(\Omega(e)) \ , \ \Omega(e^{-1}) = (\Omega(e))^{-1}.$$

Star of a vertex v is denoted by $star(v) = \{e : e \in E, i(e) = v\}$. The number of edges in $star(v)$ is called the valence (or degree) of v denoted by $d(v)$.
The mapping $\Omega : \Gamma_1 \to \Gamma_2$ is locally injective if it is injective on stars, that is $\Omega :_{starv_1} star(v_1) \to star(\Omega(v_1))$ is injective for each $v_1 \in V_1$. Similarly we may define locally surjective and locally bijective.
Let $\Omega : \Gamma_1 \to \Gamma_2$ be a mapping of 2-complex graphs. If v is a vertex of Γ such that $\Omega(\tilde{v}) = v$, then $\tilde{v}$ is said to be lie over v. Let α be a path in Γ with $i(\alpha) = v$ and suppose $\tilde{v}$ lies over v. A path $\tilde{\alpha}$ in $\tilde{\Gamma}$ is said to be a lift of α at $\tilde{v}$ if $\Omega(\tilde{\alpha}) = \alpha$.

Definition2.9 If $\Omega : \tilde{\Gamma} \to \Gamma$ is a locally bijective map and $\tilde{\Gamma}, \Gamma$ are connected 2-complex graphs, then $\tilde{\Gamma}$ is a called a covering graph (covering space) of Γ. The mapping Ω is called the covering map (covering projection).

Theorem2.10 Let $\Omega : \tilde{\Gamma} \to \Gamma$ be a mapping of 2-complex graphs. Then the following are equivalent:
i) The map Ω is locally injective.
ii) For each path α in Γ, if $\tilde{v}$ lies over $i(\alpha)$, then α has at most one lift at $\tilde{v}$.[2]

Theorem2.11 Let $\Omega : \tilde{\Gamma} \to \Gamma$ be a mapping of 2-complex graphs. Then the following are equivalent:
i) The map Ω is locally surjective.
ii) For each path α in Γ, if $\tilde{v}$ lies over $i(\alpha)$, then α has at least one lift at $\tilde{v}$.[2]

3 Main results

In this section we obtain the covering graph for all connected 2-complex graphs that obtained from semigroup presentation $S =< a, b : a = b >$.
Let $S =< a, b : a = b >$ be a semigroup presentation. In order to construct the Squire complex K(S) to obtain the diagram group of S. If $L(W) = 1$ where W is a positive word on S, then we have two possibilities vertices a, b. So the connected 2 −complex graph Γ_1 is given by the Figure 3.

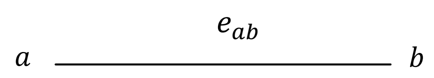


Figure 3 The connected 2-complex graph Γ_1

Note that when $L(W) = 1$, we get $2-$vertices and $1-$edge in Γ_1.
If $L(W) = 2$. In this case there are 2^2 possibilities vertices in the connected $2-$complex graph Γ_2 :

$$a^2 = aa\,, ab\,, ba \text{ and } b^2 = bb.$$

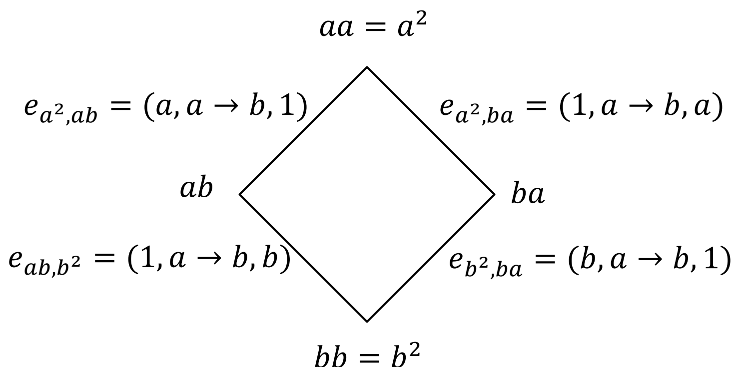


Figure 4 The connected 2-complex graph Γ_2

When $L(W) = 3$ we make two copies of Γ_2 with respect to $L(W) = 2$. So the Γ_3 in this case looks like the Figure 5.

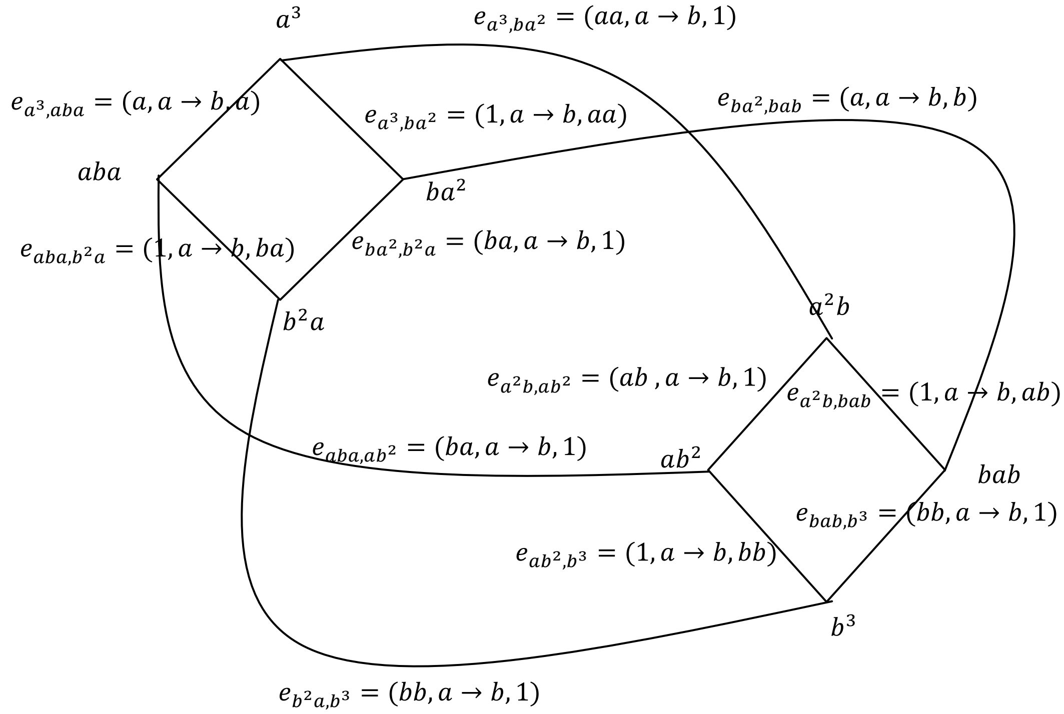


Figure 5 The connected 2-complex graph Γ_3

Note that Γ_3 is two copies of Γ_2 and each vertex in each copy are joined together respectively. Similarly, with two copies of Γ_3, we may obtain Γ_4. Repeat similar procedures for Γ_5 and so on.

Corollary 3.1 Γ_n is a connected $2-$complex graph contains 2^n vertices.

Corollary 3.2 Γ_{n+1} is two copies of Γ_n. Thus if there is e_n edges in Γ_n then the number of edges in Γ_{n+1} is $2e_n$ plus all edges between squares in Γ_{n+1}, which is 2^n.

Corollary 3.3 Vertices u and v are connected if and only if $l(u) = l(v)$.

Lemma 3.4 Vertices of Γ_n are all words of length n.

Lemma 3.5 Valance of a^n , b^n are n. The valance of $a^l b^m$ are $2n$.

Lemma 3.6 : Let $S =< a, b \ : a = b \ >$ be a semigroup presentation, then the $2-$complex graph of $D(S, W)$ is $\Gamma = \cup \Gamma_i$ where Γ_i is a connected $2-$complex graph contains all vertices of length i.

Theorem 3.7 Γ_{i+1} is the covering graph for Γ_i, for all $i \in N$.

Proof. By induction we will prove this theorem. Since we already draw $\Gamma_1, \Gamma_2, \Gamma_3$ so we have to draw Γ_i and Γ_{i+1}.
If $L(W) = k$.

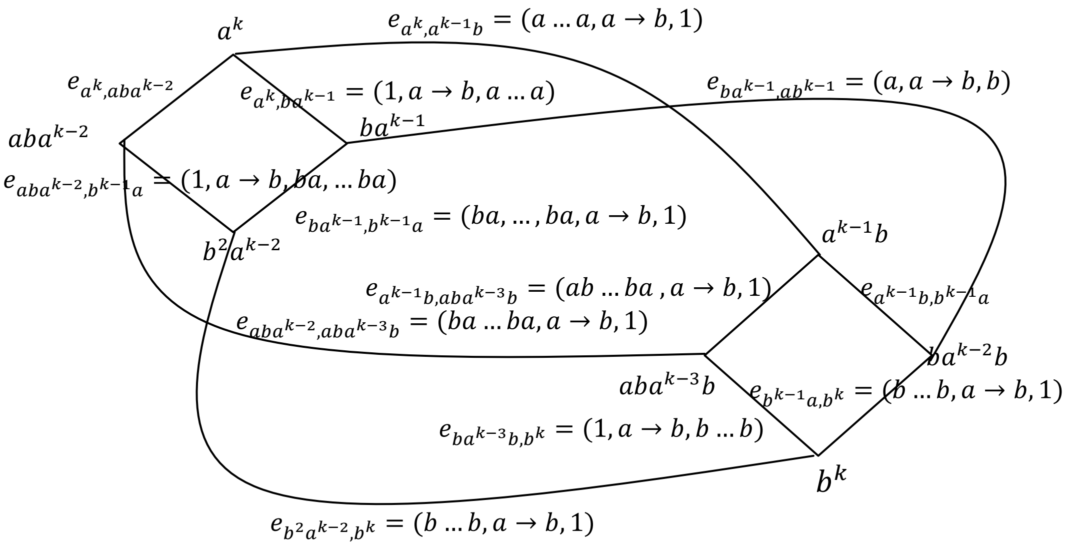


Figure 6 The connected 2-complex graph Γ_k

Finally, for $L(W) = k + 1$. In fact Γ_{k+1} is just two copies of Γ_k.

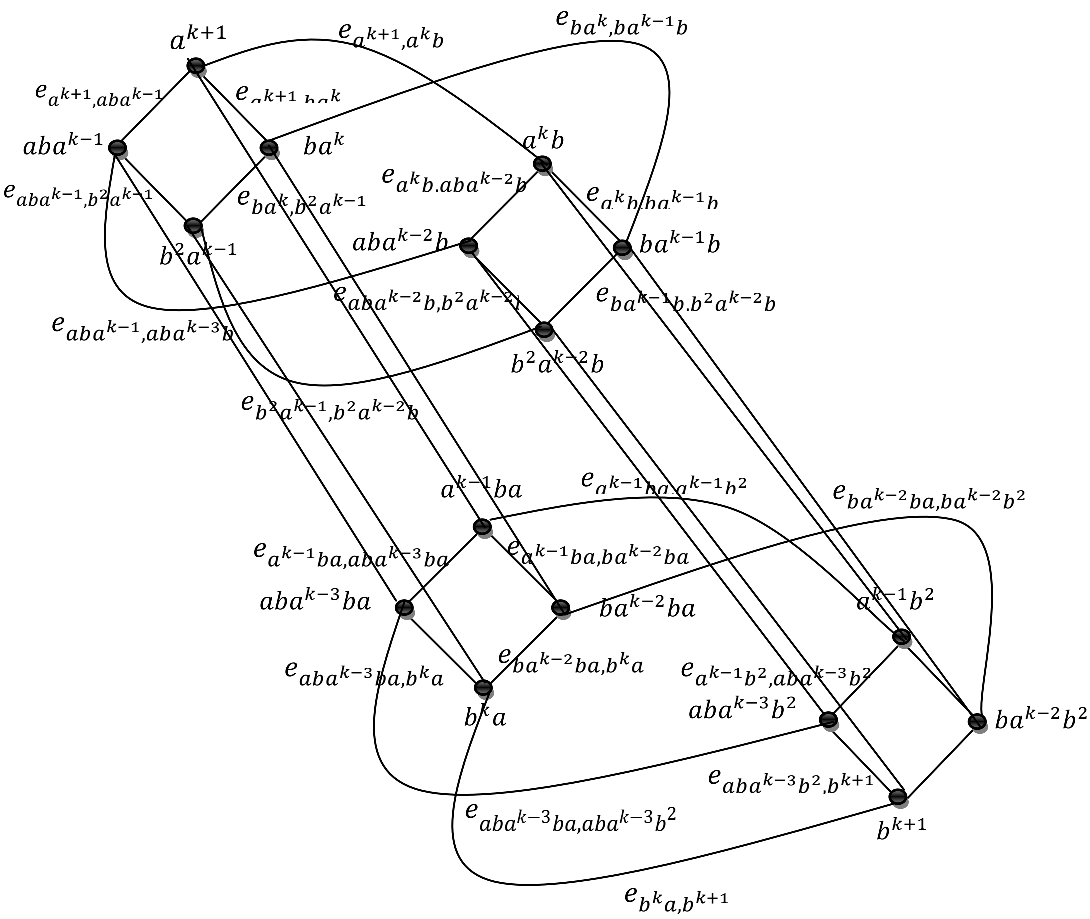


Figure 7 The connected 2-complex graph Γ_{k+1}

We will prove that Γ_{i+1} is the covering graph of Γ_i for all $i \in N$ by induction.
For $i = 1$. Our claim is to prove Γ_2 is the covering graph for Γ_1. To prove this theorem we will use theorem (2.10) and theorem (2.11).
Γ_1 and Γ_2 are connected $2-$complex graphs since if we take any two vertices in these $2-$complex graphs, we can see there is a path joining them.
Let $\Omega : \Gamma_2 \to \Gamma_1$ defined by $\Omega(a^2) = a$, $\Omega(ba) = b$, $\Omega(ab) = a$ and $\Omega(b^2) = b$. $\Omega(e_{a^2,ba}) = e_{ab}$, $\Omega(e_{ab,b^2}) = e_{ab}$, $\Omega(e_{a^2,ab}) = \phi$ and $\Omega(e_{b^2,ba}) = \phi$. We will prove that Ω is locally bijective. Choose a^2 a vertex of Γ_2 and let

$e_{a^2,ab}, e_{a^2,ba} \in$ star (a^2) and suppose $\Omega(e_{a^2,ab}) = \Omega(e_{a^2,ba}) = e_{ab}$. Then regarding e_{ab} as a path of length 1, we know that $e_{a^2,ab}$ and $e_{a^2,ba}$ are lifts of e_{ab} at a^2. Hence by theorem (2.10) $e_{a^2,ab} = e_{a^2,ba}$ that is Ω is locally injective. Now choose a^2 be any vertex of Γ_2 and $e_{ab} \in$ star (a). Regarding e_{ab} as a path of length 1, then by theorem (2.11) there exists at least one lift of this path at a^2. Such a lift is just an edge in star (a^2). Hence there is $e_{a^2,ba} \in$ star (a^2) such that $\Omega(a^2) = a$. So Ω_{a^2} : star $(a^2) \rightarrow$ star (a) is a locally surjective, and hence Ω is a locally bijective. Therefore Γ_2 is the covering graph for Γ_1. Thus our first claim is ture.

For $i = k - 1$. Assume Γ_k is the covering graph for Γ_{k-1}.

Now for $i = k$. Our claim is to prove that Γ_{k+1} is the covering graph for Γ_k. Γ_k and Γ_{k+1} are connected $2-$complex graphs since if we take any two vertices in these $2-$complex graphs, we can see there is a path joining them. So it remains to prove Γ_{k+1} is the covering graph for Γ_k. To prove that let $\Omega : \Gamma_{k+1} \rightarrow \Gamma_k$ defined by $\Omega(wx) = w$, where w is a word on a, b of length k , $x \in \{a, b\}$.

$\Omega(e_{w_1x_1,w_2x_2}) = e_{w_1,w_2}$.

Choose a^{k+1} a vertex of Γ_{k+1}, $\Omega(a^{k+1}) = a^k$ and let $e_{a^{k+1},aba^{k-1}}$, $e_{a^{k+1},a^{k-1}ba} \in$ star(a^{k+1}) such that $\Omega(e_{a^{k+1},aba^{k-1}}) = \Omega(e_{a^{k+1},a^{k-1}ba}) = e_{a^k,a^{k-1}b}$. Then regarding $e_{a^k,a^{k-1}b}$ as a path of length 1, we know that $e_{a^{k+1},aba^{k-1}}$, $e_{a^{k+1},a^{k-1}ba}$ are lifts of $e_{a^k,aba^{k-2}}$ at a^{k+1}. Hence by theorem (2.10) $e_{a^{k+1},aba^{k-1}} = e_{a^{k+1},a^{k-1}ba}$. That is Ω is a locally injective. Now choose a^{k+1} a vertex of Γ_{k+1}, and let $e_{a^k,aba^{k-2}} \in$ star(a^k). Regarding $e_{a^k,aba^{k-2}}$ as a path of length 1 , so by theorem (2.11) there exists at least one lift of this path at a^{k+1}. Such a lift is just an edge in star(a^{k+1}). Hence there is $e_{a^{k+1},aba^{k-1}} \in$ star(a^{k+1}) such that $\Omega(e_{a^{k+1},aba^{k-1}}) = e_{a^k,aba^{k-2}}$. So $\Omega_{a^{k+1}}$: star $(a^{k+1}) \rightarrow$ star (a^k) is locally surjective, and hence Ω is a locally bijective. Therefore Γ_{k+1} is the covering graph for Γ_k. The theorem is proved.■

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