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The Domination and Independence of Some Cubic Bipartite Graphs

N. Murugesan and Deepa S. Nair*

Department of Mathematics
Government Arts College, Coimbatore-18, India
*deepamtcr@gmail.com

Abstract

In this paper we discuss the relation between independent set and dominating set of finite simple graphs . In particular, we discuss them for some cubic bipartite graphs and find that the domination number is less than 1/3 of the number of vertices and independence number is half of the same.

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Keywords: Domination, Domination number, Independent set, Independence number, Cubic bipartite graph

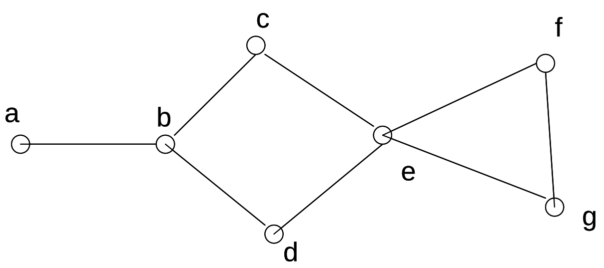
1. Introduction

Let G = (V,E) be a graph. Let D be a subset of V. If each vertex of V - D is adjacent to atleast one vertex of D, then D is called a *dominating* set in G. The domination number of a graph G denoted as γ(G) is the minimum cardinality of a dominating set in G. A minimal dominating set is a dominating set from which no vertex can be removed without destroying its dominance property. A set of vertices in a graph is said to be an independent set if no two vertices in the set are adjacent. A maximal independent set is an independent set to which no other vertex can be added to it without destroying its independence property. The number of vertices in the largest independent set of a graph G is called the independence number and is denoted by β(G).A graph is said to be a cubic graph if each of its vertices is of degree three. A bipartite graph is a graph in which vertices can be divided into two disjoint sets A and B such that every edge connects a vertex in A to one in B.

We first discuss the relation between domination number and independence number of a finite simple graph in the following section.

2. Domination and independence of Graphs

Consider the graph shown below.



In the above graph, {a, c} is an independent set.
{b, f} is a maximal independent set and also minimal dominating set.
And the domination number is 2.
{a, c, d, f } is the largest independent set and so independence number is 4.

There are some graphs for which the domination number and independence number are same. The following theorems characterize such graphs.

Theorem 2.1

An independent set of a graph G is dominating if and only if it is maximal.

Proof:-

Suppose G has n vertices. Let S be an independent set in G and also dominating set in G.
Also let $n(S) = n_1$ and $n(G - S) = n_2$. Therefore, $n = n_1 + n_2$.
Then every vertex in V – S is adjacent to at least one vertex in S.
First let us find answer for the following questions.
What will be the minimum number of edges required so as to G have an independent dominating set, and also the maximum number of edges G may have for the same.
The minimum number of edges required so as to G have an independent dominating set is n – 1. If the number of edges is < n -1, then the graph will be disconnected.

Also for a graph with a given number of vertices, maximum edges occur only when it is a complete graph. And the number of edges for a complete graph with n vertices is nC_2.
In this case, every singleton set will be independent and dominating. So the independence number and domination number is one. If S is not maximal, a vertex

can be added to it without destroying its independence property. Since S is dominating, if we add one vertex to S then the set will lose its independence property. Therefore, S should be maximal.

Conversely, assume that the independent set of a graph G is maximal.

We have to show that it is a dominating set. If the independent set does not dominate the graph, then there is at least one vertex that is neither in the set nor adjacent to any vertex in the set. Such a vertex can be added to it without destroying its independence. But this contradicts the maximality of the set.

Therefore, the set is a dominating set.

Theorem 2.2

If a simple graph G with n vertices has a vertex with degree n – 1 , then the domination number **γ**(G) is one.

Proof:-

Let G be a simple graph with n vertices. Let v be the vertex in G with degree n – 1.That is every other vertex is adjacent to v. Then {v} is a dominating set of the graph G. Hence, the cardinality of this set is the domination number and it is one.

Theorem 2.3

If G is a complete graph, then **γ**(G) = **β**(G).

Proof:-

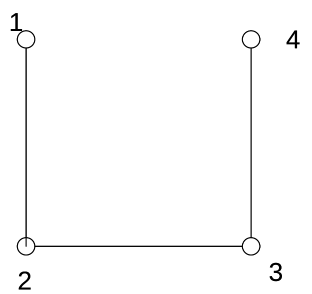
Let G be a complete graph. Then there is an edge between every pair of vertices. Therefore, each vertex is a minimal dominating set. That is, for complete graphs, **γ**(G) = 1.

Also only the singleton sets are independent. Then, the cardinality of the largest independent set is one. That is, independence number **β**(G) = 1.

But the converse of the above theorem is not true.

i.e. In a graph G, **γ**(G) = **β**(G) does not imply that G is complete.

For this, consider the graph as given below.



In the above graph, { 1, 3}, {2,4}, {1,4} are maximal independent sets as well as minimal dominating sets. No other such sets are possible.

Therefore $\gamma(G) = 2$ and $\beta(G) = 2$.
But it is not a complete graph.

Theorem 2.4

In a simple graph G, when an edge is added, the number of maximal independent sets decreases.

Proof:-

Let G be a simple graph with n vertices $v_1, v_2, \ldots, v_n$. Let S be the set of all maximal independent sets of G. Let there be two vertices v_i and v_j which are not adjacent.
Then v_i and v_j will be in some of the independent sets. Then if we join v_i and v_j, those sets would not be independent.
Therefore the number of maximal independent sets in S decreases when an edge is added in them.

Lemma 2.5

In a simple graph G, when an edge is added, the number of minimal dominating sets decreases.

Proof:-

In a graph, every maximal independent set is a dominating set.
In these sets, clearly some sets will be minimal dominating.
Consequently, by the previous theorem the number of minimal dominating sets decreases.

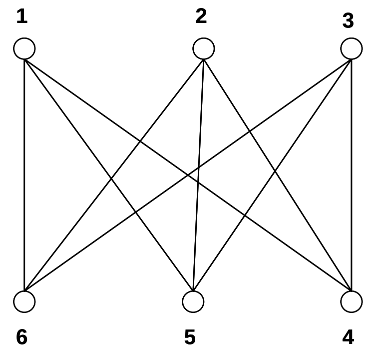
3. Domination and Independence of Cubic bipartite graphs

In this section some cubic bipartite graphs are considered and we find their domination number and independence number.

Example 3.1

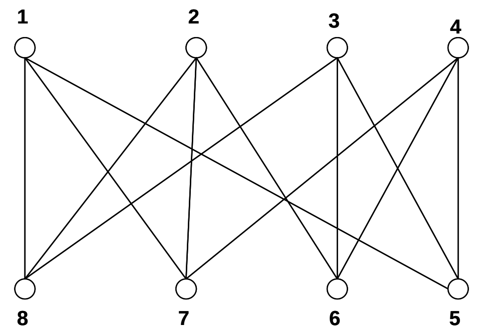
Let G be a cubic bipartite graph and n be the number of its vertices. We know that the minimum number of vertices that a cubic bipartite graph will have is 6. Also no graph that have odd number of vertices cannot be a cubic bipartite graph.
We now concentrate the cubic bipartite graphs where n = 6,8,10,12,14,16,18,20.

1. For n = 6, we have the cubic bipartite graph as given below.



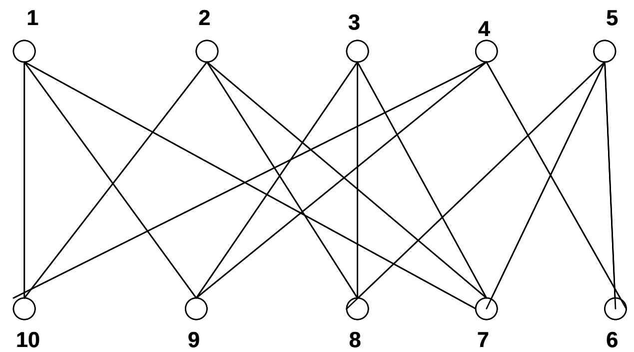
Here { 1, 5} is a dominating set. Then **γ**(G) = 2 = 6/3 = n/3.The largest independent set is { 1,2,3 }.Then **β**(G) = 3 = n /2.

For n = 8, we have the following graph,

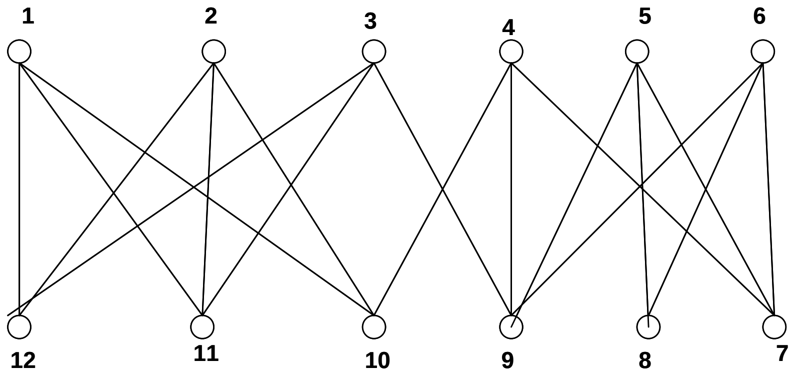


Here { 1 , 6} is a dominating set. i.e, **γ**(G) = 2 < 8/3 = n/3.
The largest independent set is { 1, 2, 3 ,4}.**β**(G) = 4 = n /2.

For n = 10, the cubic bipartite graph is as follows,

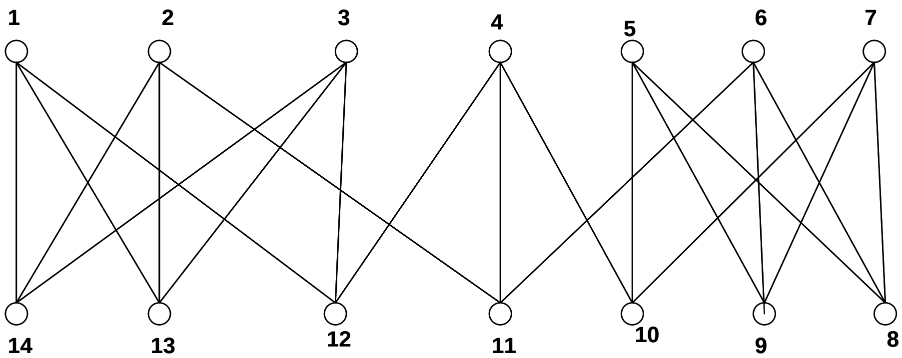


Here {2 , 5, 9} is a dominating set and hence **γ**(G) = 3 < 10/3 = n/3.
The largest independent set is {1, 2 ,3, 4,5} and hence **β**(G) = 5 = n /2.
Similarly, the following bipartite graph for n = 12,



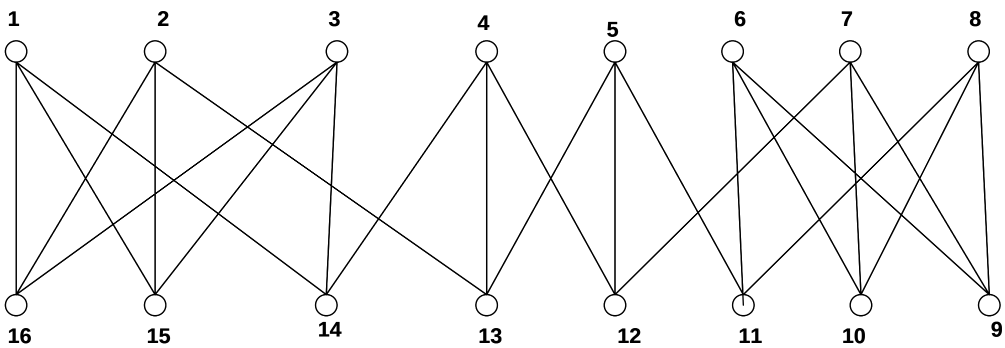
Here{1,7,9,11} is a dominating set. **γ**(G) = 4 =12/3 = n/3.
And the largest independent set is { 1,2,3,4,5,6} and **β**(G) = 6 = n /2.

For n = 14, we have,



Hence {1,5,9,11} is a dominating set. i.e, **γ**(G) = 4 < 14/3 = n/3.
Also the largest independent set is { 1,2,3,4,5,6,7} and hence **β**(G) = 7 = n /2.

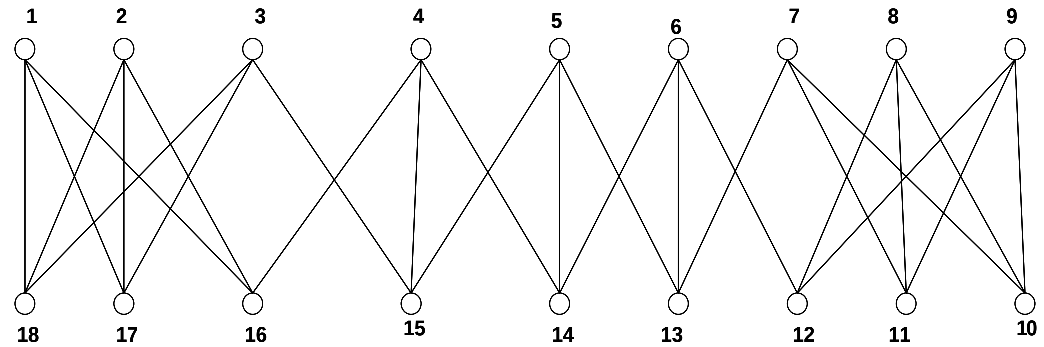
For n =16, we have



Note that {1,2,5,10,13} is a dominating set and **γ**(G) = 5 < 16/3 = n/3.

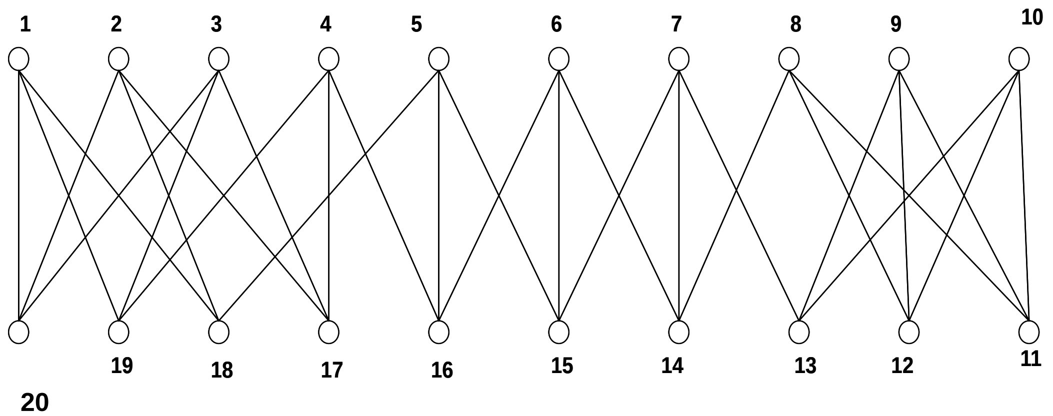
The largest independent set is {1,2,3,4,5,6,7,8} and **β**(G) = 8 = n /2.

For n = 18, we have



Here {1,6,7,11,15,16} is a dominating set. **γ**(G) = 6 = 18/3 = n/3.
The largest independent set is { 1,2,3,4,5,6,7,8,9} and **β**(G) = 9= n /2.

For n = 20, we have,



Where {1,13,14,15,16,20} is a dominating set and thus **γ**(G) = 6 < 20/3 = n/3.
The largest independent set is { 1,2,3,4,5,6,7,8,9,10}. Hence **β**(G) = 10 = n /2.

Summarizing the above examples we have the following theorems.

Theorem 3.2

For a cubic bipartite graph the domination number is less than or equal to 1/3 of the number of vertices.

Proof:-

Consider a cubic bipartite graph with n vertices. Then the set of vertices can be partitioned into two sets with each set consisting of n/2 vertices.

Also since the graph is cubic, each vertex is of degree 3. So each vertex in both of the sets will be adjacent to 3 vertices only. That is, each vertex dominates 3 vertices. So at most n/3 vertices will be enough to dominate all the other vertices.
Hence the domination number will be less than or equal to 1/3 of the number of vertices.

Theorem 3.3

For a cubic bipartite graph, the independence number is equal to 1/2 of the number of vertices.

Proof:-

Consider a cubic bipartite graph with n vertices. The vertex set can be partitioned in to two independent sets consisting of n/2 vertices each.
So the largest independent set will consist of n /2 vertices. Hence the independence number is n /2.

4. Conclusion

In section 2, some results on the domination of graphs are shown. In the above section the domination numbers of some cubic bipartite graphs are found and it is shown that those domination numbers are less than or equal to one third of the number of vertices of the graph.

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