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A Note on Optimal State Estimation for a Fractional Linear System

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Abstract

The aim this note is to consider the problem of optimal state estimation for a linear system driven by a fractional process and expressed by a fractional Langevin equation. From an approximation approach we obtained equations of the estimation for an approximate model, and proved that the true solution for the initial problem is the limit case.

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1 Introduction

Under certain situation the state of a system can influence upon its long-range behaviour. It is the case of motion in a fractal medium on which some absolutely continuous limiting distribution are supported. Also, various quantities in dynamic of financial asset prices or indexes has a long memory property.

On the other hand, not always one can directly obtain true values of these states from their system dynamics. One can only observe them via other quantities. The problem now is how to estimate the state of a system perturbated by a long memory noise.

In this note we consider a linear system described a fractional Langevin equation

$$dX_t = -bX_t dt + \sigma dB_t^H , \tag{1.1.1}$$

where $B_t^H = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$ is a fractional Brownian motion of Liouville form, H is the Hurst index $(0 < H < 1)$, $b, \sigma > 0$ are constants, and W is an ordinary standard Brownian motion.

The formal writing form (1.1.1) is understood as

$$X_t = X_0 - \int_0^t bX_s ds + \sigma B_t^H . \tag{1.1.2}$$

It is well-known that a Liouville fractional Brownian motion can be considered as the limits case of a time series of ARIMA type of long memory defined as

$$X_t = (1-L)^{-d} \Phi(L)^{-1} \Theta(L) \varepsilon_t ,$$

where (ε_t) is a sequence of centered and uncorrelated random variables of the same variance, L is the lag operator, Φ and Θ are polynomials of L with roots outside of the unit disc.

The system X_t in (1.1.2) describes a motion of particles in a liquid medium of fractal structure or the volatility of a financial asset in a financial market or merely some diffusion coefficient σ_t for various quantities in physics and technology of the form

$$dZ_t = a(t, Z_t)dt + \sigma_t dW_t .$$

The observation for the system (1.1.1) or (1.1.2) is given by a point process Y_t of the form

$$Y_t = \int_0^t h_s ds + M_t \tag{1.1.3}$$

where M_t is a Poissonian square integrable martingale and $h_s = h(X_s)$ satisfies the condition

$$E \int_0^t h_s^2 ds < \infty .$$

Basing on a fundamental result on L^2-approximation of B_t^H given in [3, 4], we will solve the problem of optimal state-estimation (1.1.1)-(1.1.3).

2 Approximation Method

It is known that $B_t^H = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$ is neither a semimartingale nor Markov process but as shown in [3, 4], B_t^H can be approximated in L^2 by semimartingale. We recall this result as follows.

For a sake of simplicity we put $\alpha = H - \frac{1}{2}$ and write from now on B_t instead of B_t^H. For every $\varepsilon > 0$ define

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}} dW_s \,. \tag{2.2.1}$$

We have the following assertions (refer to [3, 4])

Lemma 2.1. *B_t^ε is a semimartingale:*

$$dB_t^\varepsilon = \varphi(t)dt + \varepsilon^\alpha dW_t \,, \tag{2.2.2}$$

where $\varphi(t) = \alpha \int_0^t (t-s+\varepsilon)^{\alpha-1} dW_s$ is a process having absolutely continuous trajectories.

Theorem 2.2. *B_t^ε converges to B_t in $L^2(\Omega)$ when ε tends to 0. This convergence is uniform with respect to $t \in [0,T]$.*

Now we replace B_t^H in (1.1.1) by B_t^ε and we consider the approximation problem of state-estimation as follows:

The system process is described by

$$dX_t = -bX_t dt + \sigma dB_t^\varepsilon \,, \tag{2.2.3}$$

And the observation process is given as (1.1.3):

$$Y_t = \int_0^t h_s ds + M_t \tag{2.2.4}$$

After solving this problem to obtain the approximate estimation

$$\pi_t(X^\varepsilon) = \widehat{X}_t^\varepsilon = E[X_t^\varepsilon | \mathcal{F}_t^Y] \tag{2.2.5}$$

where X_t^ε is the solution of (2.2.3) we will prove that the true estimation $\pi_t = \widehat{X}_t = E[X_t | \mathcal{F}_t^Y]$ is the L^2-limit of $\pi_t(X^\varepsilon)$ when ε tends to 0.

3 Solution of (2.2.3) (refer to [3])

Substituting dB_t^ε in (2.2.3) by its expression from (2.2.2) we have

$$dX_t = -[bX_t + \sigma\varphi(t)]dt + \varepsilon^\alpha \sigma dW_t\,, \tag{3.3.1}$$

where $\varphi(t) = \alpha \int_0^t (t-s+\varepsilon)^{\alpha-1} dW_s\,.$

The equation (3.3.1) can be splitted into two equations

$$dX_1(t) = -bX_1(t)dt + \varepsilon^\alpha \sigma dW_t\,, \tag{3.3.2}$$

and

$$dX_2(t) = -bX_2(t)dt - \sigma\varphi(t)dt\,. \tag{3.3.3}$$

The solution of (3.3.1) will be $X_t = X_1(t) + X_2(t)\,.$

It is well known that (3.3.2) is a classical Langevin equation whose solution is an Ornstein-Uhlenbeck process

$$X^\varepsilon(t) = X_1^{(0)} e^{-bt} + \sigma\varepsilon^\alpha \int_0^t e^{-b(t-s)} dW_s\,, \tag{3.3.4}$$

where $X_1^{(0)}$ is the initial value of $X_1(t)$: $X_1^{(0)} = X_1(0)$ that is supposed to be a random variable independent of $(W_t, 0 \le t \le T)$.

The equation (3.3.3) is an ordinary differential equation for every fixed ω and its solution is

$$X_2^\varepsilon(t) = X_2^{(0)} e^{-bt} - \sigma \int_0^t e^{-b(t-s)} \varphi(s) dW_s\,. \tag{3.3.5}$$

where $X_2^{(0)} = X_2(0)$ independent of $(W_t, 0 \le t \le T)$ and $X_1^{(0)} + X_2^{(0)}$ is the initial value of $X_t : X_0 = X_1^{(0)} + X_2^{(0)}\,.$

Now the solution (3.3.1) is

$$X_t^\varepsilon = X_1^\varepsilon(t) + X_2^\varepsilon(t)\,. \tag{3.3.6}$$

4 Solution of (1.1.2)

We will show that approximate solution X_t^ε converges to the solution X_t of (1.1.2). Indeed,

$$X_t - X_t^\varepsilon = -b \int_0^t (X_s - X_s^\varepsilon) ds + \sigma(B_t - B_t^\varepsilon)\,, \tag{4.4.1}$$

hence

$$\|X_t - X_t^\varepsilon\| = \|b\int_0^t (X_s - X_s^\varepsilon)ds\| + \sigma\|(B_t - B_t^\varepsilon)\|\,, \tag{4.4.2}$$

where $\|.\|$ stands for the norm in $L^2(\Omega)$, and it follows from the proof of the convergence $B_t^\varepsilon \xrightarrow{L^2} B_t$ in [4] that

$$\|(B_t - B_t^\varepsilon)\| \leq C(\alpha)\varepsilon^{\alpha+\frac{1}{2}} \tag{4.4.3}$$

where $C(\alpha)$ depends only on α.

Applying Gronwall's lemma to (4.4.2) yields

$$\|X_t - X_t^\varepsilon\| \leq \sigma C(\alpha)\varepsilon^{\alpha+\frac{1}{2}}e^{bt} \tag{4.4.4}$$

and therefore

$$\sup_{0\leq t\leq T}\|X_t - X_t^\varepsilon\| \leq \sigma C(\alpha)\varepsilon^{\alpha+\frac{1}{2}}e^{bt}\,,$$

So $X_t^\varepsilon \to X_t$ in $L^2(\Omega)$ uniformly with respect to $t \in [0, T]$.

5 State-estimation for X_t^ε

We return back to the approximation problem of state estimation:

$$dX_t = -bX_t dt + \sigma dB_t^\varepsilon\,, \tag{5.5.1}$$

$$Y_t = \int_0^t h_s ds + M_t \tag{5.5.2}$$

and we see that

$$E[X_t^\varepsilon|\mathcal{F}_t^Y] = E[X_1^\varepsilon(t) + X_2^\varepsilon(t)|\mathcal{F}_t^Y] = E[X_1^\varepsilon(t)|\mathcal{F}_t^Y] + E[X_2^\varepsilon(t)|\mathcal{F}_t^Y]\,,$$

So

$$\pi_t(X^\varepsilon) = \pi_t(X_1^\varepsilon) + \pi_t(X_2^\varepsilon) \tag{5.5.3}$$

5.1 Equation for $\pi_t(X_1^\varepsilon)$

Note again that $X_1^\varepsilon(t)$ is an Ornstein-Uhlenbeck process satisfying the classical Langevin equation

$$dX_1^\varepsilon(t) = -bX_1^\varepsilon(t)dt + adW_t \tag{5.5.4}$$

where $a = \sigma\varepsilon^\alpha > 0$. According to a result in [1] on the state estimation from point process observation we can write the estimation equation for $\pi_t(X_1^\varepsilon) = \widehat{X_1^\varepsilon(t)}$ as follows

$$\begin{aligned}\pi_t(X_1^\varepsilon) = \pi_0(X_1^\varepsilon)e^{-bt} - \int_0^t \Big[b\,\pi_s(X_1^\varepsilon)\Big]ds \\ + \int_0^t \big[\pi_s(h)\big]^{-1}\big[\pi_s(hX_1^\varepsilon) - \pi_s(X_1^\varepsilon)\pi_s(h)\big]\big(dY_s - \pi_s(h)ds\big)\end{aligned} \tag{5.5.5}$$

where $\pi_s(h) = E[h(X_s)|\mathcal{F}_s^Y]$.

5.2 Equation for $\pi_t(X_2^\varepsilon) = \widehat{X_2^\varepsilon(t)} = E[X_2^\varepsilon(t)|\mathcal{F}_t^Y]$

We know that $X_2^\varepsilon(t)$ is the solution of (3.3.3) that is

$$dX_2^\varepsilon(t) = -bX_2^\varepsilon(t)dt - \sigma\varphi(t)dt\,.$$

or

$$X_2^\varepsilon(t) = X_2^{(0)}e^{-bt} - \sigma\int_0^t e^{-b(t-s)}\varphi(s)dW_s\,.$$

Hence

$$\widehat{X_2^\varepsilon(t)} = \widehat{X_2^{(0)}}e^{-bt} - \sigma\int_0^t e^{-b(t-s)}\widehat{\varphi(s)}dW_s$$

or

$$\pi_t(X_2^\varepsilon) = \pi_0(X_2^\varepsilon)e^{-bt} - \sigma\int_0^t e^{-b(t-s)}\pi_s(\varphi)dW_s \tag{5.5.6}$$

where $\pi_s(\varphi) = \widehat{\varphi} = E[\varphi(s)|\mathcal{F}_s^Y]$. Combining (5.5.3), (5.5.5) and (5.5.6) we have

5.3 Theorem 5.1

The state estimation for X_t^ε is given by the following equation:

$$\pi_t(X^\varepsilon) = \pi_0(X^\varepsilon)e^{-bt} - \int_0^t \left[b\,\pi_s(X_1^\varepsilon) + \sigma e^{-b(t-s)}\pi_s(\varphi)\right]ds$$
$$+ \int_0^t \left[\pi_s(h)\right]^{-1}\left[\pi_s(hX_1^\varepsilon) - \pi_s(X_1^\varepsilon)\pi_s(h)\right]\left(dY_s - \pi_s(h)ds\right). \quad (5.5.7)$$

We are now in the position to get the estimation of the fractional Ornstein-Uhlenbeck process X_t given by (1.1.2), based on observation given by (1.1.3). We see from the Jensen inequality that

$$\begin{aligned}\|\pi_t(X) - \pi_t(X^\varepsilon)\|^2 &= \|E[X_t|\mathcal{F}_t^Y] - E[X_t^\varepsilon|\mathcal{F}_t^Y]\|^2 \\ &= \|E[(X_t - X_t^\varepsilon)|\mathcal{F}_t^Y]\|^2 = E\big\{E[(X_t - X_t^\varepsilon)|\mathcal{F}_t^Y]^2\big\} \\ &\le E\big\{E[(X_t - X_t^\varepsilon)^2|\mathcal{F}_t^Y]\big\} = E[(X_t - X_t^\varepsilon)^2] = \|X_t - X_t^\varepsilon\|^2 \to 0\end{aligned}$$

when $\varepsilon \to 0$ as mentioned in (4.4.4).

6 State-estimation π_t

It follows from all what presented in Section 5 that we have

Theorem 5.2. *The optimal state estimation π_t of the fractional process X_t given by*

$$dX_t = -bX_t dt + \sigma dB_t\,,$$

from the point observation Y_t given by

$$Y_t = \int_0^t h_s ds + M_t$$

is the L^2-limit of π_t^ε given by (5.5.7) as $\varepsilon \to 0$.

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