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On the Spectral Norms of r-Circulant Matrices with the k-Fibonacci and k-Lucas Numbers[1]

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Abstract

In this paper, we consider the k-Fibonacci and k-Lucas sequences $\{F_{k,n}\}_{n\in N}$ and $\{L_{k,n}\}_{n\in N}$. Let $\mathcal{A} = C_r(F_{k,0}, F_{k,1}, \cdots, F_{k,n-1})$ and $\mathcal{B} = C_r(L_{k,0}, L_{k,1}, \cdots, L_{k,n-1})$ be r-circulant matrices. Afterwards, we give upper and lower bounds for the spectral norms of matrices $\mathcal{A}$ and $\mathcal{B}$. In addition, we obtain some bounds for the spectral norms of Hadamard and Kronecker products of these matrices.

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1 Introduction and Preliminaries

For $n \geq 1$, let k be any positive real number, then the k-Fibonacci sequence $\{F_{k,n}\}_{n\in N}$ and the k-Lucas sequence $\{L_{k,n}\}_{n\in N}$ are defined respectively by the following equations:

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad F_{k,0} = 0, \quad F_{k,1} = 1$$

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1}, \quad L_{k,0} = 2, \quad L_{k,1} = k$$

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Obviously, when $k = 1$, these two sequences reduce to the well-known Fibonacci sequence $\{F_n\}_{n\in N}$ and Lucas sequence $\{L_n\}_{n\in N}$, respectively.

Let α and β be the roots of the characteristic equation $x^2 - kx - 1 = 0$, then the Binet formulas of the sequences $\{F_{k,n}\}_{n\in N}$ and $\{L_{k,n}\}_{n\in N}$ have the form

$$F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_{k,n} = \alpha^n + \beta^n$$

Recently, some authors have given generating functions, derivation of sums and combinatorial representations of the k-Fibonacci numbers and its certain generalizations [1-4]. For example, Kilic [2] has given the sums of squares of the terms of sequence $\{u_n\}$ and the sums of products of consecutive terms of $\{u_n\}$, then he has obtained the generating functions and combinatorial representations of the products $u_n u_{n+1}$ and $u_n u_{n+2}$. Falcon and Plaza [3] have derived the generating functions and sums of the k-Fibonacci sequence $\{F_{k,an+r}\}_{n\in N}$ and the alternating sequence $\{(-1)^n F_{k,an+r}\}_{n\in N}$, where a, r are integers and $0 \le r \le a-1$.

Further, there have been several papers on the norms of some special matrices [5-11]. For example, Solak and Bozkurt [5] have found out upper and lower bounds for the spectral norms of Cauchy-Toeplitz and Cauchy-Hankel matrices in the forms $T_n = [\frac{1}{a+(i-j)b}]_{i,j=1}^n$, $H_n = [\frac{1}{a+(i+j)b}]_{i,j=1}^n$. Solak [7,8] has defined $A = [a_{ij}]$ and $B = [b_{ij}]$ as $n\times n$ circulant matrices, where $a_{ij} \equiv F_{(mod(j-i,n))}$ and $b_{ij} \equiv L_{(mod(j-i,n))}$, then he has given some bounds for the A and B matrices concerned with the spectral and Euclidean norms. Bani-Domi and Kittaneh [11] have established two general norm equalities for circulant and skew circulant operator matrices, furthermore, they also have obtained pinching type inequalities for operator matrices.

In this paper, let $\mathcal{A} = C_r(F_{k,0}, F_{k,1}, \cdots, F_{k,n-1})$ and $\mathcal{B} = C_r(L_{k,0}, L_{k,1}, \cdots, L_{k,n-1})$ be r-circulant matrices. Afterwards, we give upper and lower bounds for the spectral norms of matrices $\mathcal{A}$ and $\mathcal{B}$. In the partial case $k = 1$, we find out lower and upper bounds for the spectral norms of r-circulant matrices with the Fibonacci and Lucas numbers. In addition, we obtain some bounds for the spectral norms of Hadamard and Kronecker products of these matrices.

Now we give some preliminaries related to our study. A matrix $C = [c_{ij}] \in M_{n,n}(\mathcal{C})$ is called a r-circulant matrix if it is of the form

$$c_{ij} = \begin{cases} c_{j-i}, & j \ge i \\ rc_{n+j-i}, & j < i \end{cases}$$

Obviously, the r-circulant matrix C is determined by parameter r and its first row elements $c_0, c_1, \cdots, c_{n-1}$, thus we denote $C = C_r(c_0, c_1, \cdots, c_{n-1})$. Especially, let $r = 1$, the matrix C is called a circulant matrix.

For any $A = [a_{ij}] \in M_{m,n}(\mathcal{C})$. The well-known Frobenius (or Euclidean) norm of matrix A is

$$\|A\|_F = \left[\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right]^{\frac{1}{2}}$$

and also the spectral norm of matrix A is

$$\|A\|_2 = \sqrt{\max_{1\le i\le n}\lambda_i(A^H A)}$$

where $\lambda_i(A^H A)$ is eigenvalue of $A^H A$ and A^H is conjugate transpose of matrix A. Then the following inequality holds:

$$\frac{1}{\sqrt{n}}\|A\|_F \le \|A\|_2 \le \|A\|_F \tag{1}$$

Lemma 1[12] *For any $A, B \in M_{m,n}(\mathcal{C})$, we have*

$$\|A \circ B\|_2 \le \|A\|_2\|B\|_2$$

where $A \circ B$ is the Hadamard product of A and B.

Lemma 2[12] *Let $A \in M_{m,n}(\mathcal{C})$, $B \in M_{p,q}(\mathcal{C})$ be given, then we have*

$$\|A \otimes B\|_2 = \|A\|_2\|B\|_2$$

where $A \otimes B$ is the Kronecker product of A and B.

Lemma 3[2] *Let $F_{k,n}$ be the n-th term of the sequence $\{F_{k,n}\}_{n\in N}$, then we have*

$$\sum_{i=0}^{n} F_{k,i}^2 = \frac{F_{k,n+1}F_{k,n}}{k} \tag{2}$$

Lemma 4 *For $n \ge 1$, then we have the following recursion formulas*
(i) $L_{k,n} = kF_{k,n} + 2F_{k,n-1}$
(ii)$L_{k,n}L_{k,n-1} = (k^2+4)F_{k,n}F_{k,n-1} + (-1)^{n-1}\cdot 2k$.

Proof: (i) Since $F_{k,n} = \frac{\alpha^n-\beta^n}{\alpha-\beta}$ and $\alpha+\beta = k$, then we have

$$\begin{aligned} kF_{k,n} + 2F_{k,n-1} &= 2F_{k,n+1} - kF_{k,n} \\ &= \frac{2}{\alpha-\beta}(\alpha^{n+1}-\beta^{n+1}) - \frac{k}{\alpha-\beta}(\alpha^n-\beta^n) \\ &= \frac{1}{\alpha-\beta}[\alpha^n(2\alpha-k) - \beta^n(2\beta-k)] \\ &= \alpha^n+\beta^n = L_{k,n} \end{aligned}$$

(ii) Taking into account $\alpha+\beta = k$ and $\alpha\beta = -1$, then we have

$$\begin{aligned} F_{k,n}F_{k,n-1} &= \frac{\alpha^n-\beta^n}{\alpha-\beta}\cdot\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta} = \frac{\alpha^{2n-1}+\beta^{2n-1}-(\alpha\beta)^{n-1}(\alpha+\beta)}{(\alpha-\beta)^2} \\ &= \frac{\alpha^{2n-1}+\beta^{2n-1}-(-1)^{n-1}k}{k^2+4} \end{aligned}$$

then we obtain

$$\alpha^{2n-1}+\beta^{2n-1}=(k^2+4)F_{k,n}F_{k,n-1}+(-1)^{n-1}k$$

hence

$$\begin{aligned} L_{k,n}L_{k,n-1} &= (\alpha^n+\beta^n)\cdot(\alpha^{n-1}+\beta^{n-1})=\alpha^{2n-1}+\beta^{2n-1}+(-1)^{n-1}k \\ &= (k^2+4)F_{k,n}F_{k,n-1}+(-1)^{n-1}\cdot 2k \end{aligned}$$

Thus, the proof is completed.

2 Main Results

Theorem 1 *Let $\mathcal{A}=C_r(F_{k,0},F_{k,1},\cdots,F_{k,n-1})$ be r-circulant matrix, where $r\in\mathcal{C}$.*

(i) If $|r|\ge 1$, then

$$\sqrt{\frac{F_{k,n}F_{k,n-1}}{k}}\le\|\mathcal{A}\|_2\le\frac{|r|-|r|^n(F_{k,n}+|r|F_{k,n-1})}{1-k|r|-|r|^2}$$

(ii)If $|r|<1$, then

$$|r|\sqrt{\frac{F_{k,n}F_{k,n-1}}{k}}\le\|\mathcal{A}\|_2\le\frac{F_{k,n}+F_{k,n-1}-1}{k}.$$

Proof: The matrix $\mathcal{A}$ is of the form

$$\mathcal{A}=\begin{pmatrix} F_{k,0} & F_{k,1} & F_{k,2} & \cdots & F_{k,n-1} \\ rF_{k,n-1} & F_{k,0} & F_{k,1} & \cdots & F_{k,n-2} \\ rF_{k,n-2} & rF_{k,n-1} & F_{k,0} & \cdots & F_{k,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rF_{k,1} & rF_{k,2} & rF_{k,3} & \cdots & F_{k,0} \end{pmatrix}$$

then we have

$$\|\mathcal{A}\|_F^2=\sum_{i=0}^{n-1}(n-i)F_{k,i}^2+\sum_{i=1}^{n-1}i|r|^2F_{k,i}^2$$

when $|r|\ge 1$, by Lemma 3, we obtain

$$\|\mathcal{A}\|_F^2\ge\sum_{i=0}^{n-1}(n-i)F_{k,i}^2+\sum_{i=1}^{n-1}iF_{k,i}^2=n\sum_{i=0}^{n-1}F_{k,i}^2=n\cdot\frac{F_{k,n}F_{k,n-1}}{k}$$

hence

$$\|\mathcal{A}\|_2\ge\frac{1}{\sqrt{n}}\|\mathcal{A}\|_F\ge\sqrt{\frac{F_{k,n}F_{k,n-1}}{k}}$$

when $|r|<1$, we also obtain

$$\|\mathcal{A}\|_F^2 \geq \sum_{i=0}^{n-1}(n-i)|r|^2F_{k,i}^2+\sum_{i=1}^{n-1}i|r|^2F_{k,i}^2=n|r|^2\sum_{i=0}^{n-1}F_{k,i}^2=n\cdot\frac{|r|^2F_{k,n}F_{k,n-1}}{k}$$

hence

$$\|\mathcal{A}\|_2 \geq \frac{1}{\sqrt{n}}\|\mathcal{A}\|_F \geq |r|\sqrt{\frac{F_{k,n}F_{k,n-1}}{k}}$$

On the other hand, let $f(x)=\sum\limits_{i=0}^{n-1}F_{k,i}x^i$ be a scalar-valued polynomial, and $\pi_r=C_r(0,1,0,\cdots,0)$ be a r-circulant matrix. then we have

$$\mathcal{A}=f(\pi_r)=\sum_{i=0}^{n-1}F_{k,i}\pi_r^i$$

hence

$$\|\mathcal{A}\|_2=\|\sum_{i=0}^{n-1}F_{k,i}\pi_r^i\|_2 \leq \sum_{i=0}^{n-1}\|F_{k,i}\pi_r^i\|_2 \leq \sum_{i=0}^{n-1}F_{k,i}\|\pi_r\|_2^i$$

Since the matrix $\pi_r^H\pi_r$ is of the form

$$\pi_r^H\pi_r=\begin{pmatrix} |r|^2 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{pmatrix}$$

hence

$$\|\pi_r\|_2=\sqrt{\max_{1\leq i\leq n}\lambda_i(\pi_r^H\pi_r)}=\begin{cases}|r|, & |r|\geq 1\\ 1, & |r|<1\end{cases}$$

when $|r|\geq 1$, $1-\alpha|r|\neq 0$ and $1-\beta|r|\neq 0$, so we have

$$\begin{aligned}\|\mathcal{A}\|_2 &\leq \sum_{i=0}^{n-1}F_{k,i}|r|^i=\sum_{i=0}^{n-1}|r|^i\frac{\alpha^i-\beta^i}{\alpha-\beta}=\frac{1}{\alpha-\beta}\Big[\frac{1-(\alpha|r|)^n}{1-\alpha|r|}-\frac{1-(\beta|r|)^n}{1-\beta|r|}\Big]\\ &= \frac{(\alpha-\beta)|r|-(\alpha^n-\beta^n)|r|^n+\alpha\beta|r|^{n+1}(\alpha^{n-1}-\beta^{n-1})}{(\alpha-\beta)(1-(\alpha+\beta)|r|+\alpha\beta|r|^2)}\\ &= \frac{|r|-|r|^n(F_{k,n}+|r|F_{k,n-1})}{1-k|r|-|r|^2}.\end{aligned}$$

when $|r|<1$, similarly, we have

$$\|\mathcal{A}\|_2 \leq \sum_{i=0}^{n-1}F_{k,i}=\frac{F_{k,n}+F_{k,n-1}-1}{k}$$

Thus, the proof is completed.

If we choose $k=1$ in Theorem 1, then we have the following result:

Corollary 1 *Let $\mathcal{A}=C_r(F_0,F_1,\cdots,F_{n-1})$ be r-circulant matrix, where $r\in\mathcal{C}$, and F_n is the n-th Fibonacci number.*

(i) If $|r|\geq 1$, then

$$\sqrt{F_nF_{n-1}}\leq\|\mathcal{A}\|_2\leq\frac{|r|-|r|^n(F_n+|r|F_{n-1})}{1-|r|-|r|^2}$$

(ii)If $|r|<1$, then

$$|r|\sqrt{F_nF_{n-1}}\leq\|\mathcal{A}\|_2\leq F_{n+1}-1.$$

In fact, this Corollary gives lower and upper bounds for the spectral norm of r-circulant matrix with the Fibonacci numbers.

Theorem 2 *Let $\mathcal{B}=C_r(L_{k,0},L_{k,1},\cdots,L_{k,n-1})$ be r-circulant matrix, where $r\in\mathcal{C}$.*

(i) If $|r|\geq 1$, then

$$\sqrt{(k+\frac{4}{k})F_{k,n}F_{k,n-1}+2(1+(-1)^{n-1})}\leq\|\mathcal{B}\|_2$$

$$\leq\frac{2-k|r|-|r|^n[(k+2|r|)F_{k,n}+(2-k|r|)F_{k,n-1}]}{1-k|r|-|r|^2}$$

(ii)If $|r|<1$, then

$$|r|\sqrt{(k+\frac{4}{k})F_{k,n}F_{k,n-1}+2(1+(-1)^{n-1})}\leq\|\mathcal{B}\|_2$$

$$\leq\frac{(k+2)F_{k,n}+(2-k)F_{k,n-1}+k-2}{k}.$$

Proof: The matrix $\mathcal{B}$ is of the form

$$\mathcal{B}=\begin{pmatrix} L_{k,0} & L_{k,1} & L_{k,2} & \cdots & L_{k,n-1}\\ rL_{k,n-1} & L_{k,0} & L_{k,1} & \cdots & L_{k,n-2}\\ rL_{k,n-2} & rL_{k,n-1} & L_{k,0} & \cdots & L_{k,n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ rL_{k,1} & rL_{k,2} & rL_{k,3} & \cdots & L_{k,0}\end{pmatrix}$$

then we have

$$\|\mathcal{B}\|_F^2=\sum_{i=0}^{n-1}(n-i)L_{k,i}^2+\sum_{i=1}^{n-1}i|r|^2L_{k,i}^2$$

since

$$\begin{aligned} L_{k,n}L_{k,n-1} &= (kL_{k,n-1}+L_{k,n-2})L_{k,n-1} = kL_{k,n-1}^2 + L_{k,n-1}L_{k,n-2} = \cdots \\ &= kL_{k,n-1}^2 + kL_{k,n-2}^2 + \cdots + kL_{k,1}^2 + L_{k,1}L_{k,0} \\ &= k\sum_{i=0}^{n-1} L_{k,i}^2 - 2k \end{aligned}$$

then we have

$$\sum_{i=0}^{n-1} L_{k,i}^2 = \frac{L_{k,n}L_{k,n-1}}{k} + 2 \qquad (3)$$

when $|r| \geq 1$, from (3), we obtain

$$\|\mathcal{B}\|_F^2 \geq \sum_{i=0}^{n-1}(n-i)L_{k,i}^2 + \sum_{i=1}^{n-1} iL_{k,i}^2 = n\sum_{i=0}^{n-1} L_{k,i}^2 = n\cdot\left(\frac{L_{k,n}L_{k,n-1}}{k}+2\right)$$

hence, by Lemma 4, we have

$$\|\mathcal{B}\|_2 \geq \frac{1}{\sqrt{n}}\|\mathcal{B}\|_F \geq \sqrt{\frac{L_{k,n}L_{k,n-1}}{k}+2} = \sqrt{(k+\frac{4}{k})F_{k,n}F_{k,n-1} + 2(1+(-1)^{n-1})}$$

when $|r| < 1$, we also obtain

$$\|\mathcal{B}\|_F^2 \geq \sum_{i=0}^{n-1}(n-i)|r|^2L_{k,i}^2 + \sum_{i=1}^{n-1} i|r|^2L_{k,i}^2 = n|r|^2\sum_{i=0}^{n-1} L_{k,i}^2 = n|r|^2\cdot\left(\frac{L_{k,n}L_{k,n-1}}{k}+2\right)$$

hence

$$\|\mathcal{B}\|_2 \geq \frac{1}{\sqrt{n}}\|\mathcal{B}\|_F \geq |r|\sqrt{(k+\frac{4}{k})F_{k,n}F_{k,n-1} + 2(1+(-1)^{n-1})}$$

On the other hand, let $g(x) = \sum_{i=0}^{n-1} L_{k,i}x^i$ be a scalar-valued polynomial, and $\pi_r = C_r(0,1,0,\cdots,0)$ be a r-circulant matrix. then we have

$$\mathcal{B} = g(\pi_r) = \sum_{i=0}^{n-1} L_{k,i}\pi_r^i$$

hence

$$\|\mathcal{B}\|_2 = \|\sum_{i=0}^{n-1} L_{k,i}\pi_r^i\|_2 \leq \sum_{i=0}^{n-1}\|L_{k,i}\pi_r^i\|_2 \leq \sum_{i=0}^{n-1} L_{k,i}\|\pi_r\|_2^i$$

while

$$\|\pi_r\|_2 = \sqrt{\max_{1\leq i\leq n}\lambda_i(\pi_r^H\pi_r)} = \begin{cases} |r|, & |r| \geq 1 \\ 1, & |r| < 1 \end{cases}$$

hence, when $|r| \geq 1$, by Lemma 4, we have

$$\begin{aligned}
\|\mathcal{B}\|_2 &\leq \sum_{i=0}^{n-1} L_{k,i}|r|^i = \sum_{i=0}^{n-1} |r|^i(\alpha^i+\beta^i) = \frac{1-(\alpha|r|)^n}{1-\alpha|r|} + \frac{1-(\beta|r|)^n}{1-\beta|r|} \\
&= \frac{2-(\alpha+\beta)|r|-(\alpha^n+\beta^n)|r|^n+\alpha\beta|r|^{n+1}(\alpha^{n-1}+\beta^{n-1})}{1-(\alpha+\beta)|r|+\alpha\beta|r|^2} \\
&= \frac{2-k|r|-|r|^n(L_{k,n}+|r|L_{k,n-1})}{1-k|r|-|r|^2} \\
&= \frac{2-k|r|-|r|^n[(k+2|r|)F_{k,n}+(2-k|r|)F_{k,n-1}]}{1-k|r|-|r|^2}.
\end{aligned}$$

when $|r| < 1$, similarly, we have

$$\|\mathcal{B}\|_2 \leq \sum_{i=0}^{n-1} L_{k,i} = \frac{(k+2)F_{k,n}+(2-k)F_{k,n-1}+k-2}{k}$$

Thus, the proof is completed.

When $k=1$ in Theorem 2, then we have the following result:

Corollary 2 *Let $\mathcal{B} = C_r(L_0, L_1, \cdots, L_{n-1})$ be r-circulant matrix, where $r \in \mathcal{C}$, and L_n is the n-th Lucas number.*

(i) If $|r| \geq 1$, then

$$\sqrt{5F_nF_{n-1}+2(1+(-1)^{n-1})} \leq \|\mathcal{B}\|_2 \leq \frac{2-|r|-|r|^n[(1+2|r|)F_n+(2-|r|)F_{n-1}]}{1-|r|-|r|^2}$$

(ii)If $|r| < 1$, then

$$|r|\sqrt{5F_nF_{n-1}+2(1+(-1)^{n-1})} \leq \|\mathcal{B}\|_2 \leq 3F_n+F_{n-1}-1.$$

In fact, this Corollary gives lower and upper bounds for the spectral norm of r-circulant matrix with the Lucas numbers.

Considering the results of Theorem 1 and Theorem 2, then we have the following important results.

Corollary 3 *Let $\mathcal{A} = C_r(F_{k,0}, F_{k,1}, \cdots, F_{k,n-1})$ and $\mathcal{B} = C_r(L_{k,0}, L_{k,1}, \cdots, L_{k,n-1})$ be r-circulant matrices, where $r \in \mathcal{C}$.*

(i) If $|r| \geq 1$, then

$$\begin{aligned}
\|\mathcal{A}\circ\mathcal{B}\|_2 \leq\ & \frac{|r|-|r|^n(F_{k,n}+|r|F_{k,n-1})}{1-k|r|-|r|^2} \\
&\times\frac{2-k|r|-|r|^n[(k+2|r|)F_{k,n}+(2-k|r|)F_{k,n-1}]}{1-k|r|-|r|^2}
\end{aligned}$$

(ii)If $|r| < 1$*, then*

$$\|\mathcal{A} \circ \mathcal{B}\|_2 \leq \frac{(F_{k,n} + F_{k,n-1} - 1) \times [(k+2)F_{k,n} + (2-k)F_{k,n-1} + k - 2]}{k^2}.$$

Proof: Since $\|\mathcal{A} \circ \mathcal{B}\|_2 \leq \|\mathcal{A}\|_2 \|\mathcal{B}\|_2$, the proof is trivial by Theorems 1 and 2.

Corollary 4 *Let* $\mathcal{A} = C_r(F_{k,0}, F_{k,1}, \cdots, F_{k,n-1})$ *and* $\mathcal{B} = C_r(L_{k,0}, L_{k,1}, \cdots, L_{k,n-1})$ *be r-circulant matrices, where* $r \in \mathcal{C}$*.*

(i) If $|r| \geq 1$*, then*

$$\begin{aligned}\|\mathcal{A} \otimes \mathcal{B}\|_2 \;\; \leq \;\; & \frac{|r| - |r|^n (F_{k,n} + |r| F_{k,n-1})}{1 - k|r| - |r|^2} \\ & \times \frac{2 - k|r| - |r|^n [(k + 2|r|) F_{k,n} + (2 - k|r|) F_{k,n-1}]}{1 - k|r| - |r|^2}\end{aligned}$$

and

$$\|\mathcal{A} \otimes \mathcal{B}\|_2 \geq \frac{1}{k} \sqrt{F_{k,n} F_{k,n-1} [(k^2+4) F_{k,n} F_{k,n-1} + 2k(1 + (-1)^{n-1})]}$$

(ii)If $|r| < 1$*, then*

$$\|\mathcal{A} \otimes \mathcal{B}\|_2 \leq \frac{(F_{k,n} + F_{k,n-1} - 1) \times [(k+2)F_{k,n} + (2-k)F_{k,n-1} + k - 2]}{k^2}$$

and

$$\|\mathcal{A} \otimes \mathcal{B}\|_2 \geq \frac{|r|^2}{k} \sqrt{F_{k,n} F_{k,n-1} [(k^2+4) F_{k,n} F_{k,n-1} + 2k(1 + (-1)^{n-1})]}.$$

Proof: Since $\|\mathcal{A} \otimes \mathcal{B}\|_2 = \|\mathcal{A}\|_2 \|\mathcal{B}\|_2$, the proof is trivial by Theorems 1 and 2.

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