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On Quasi-Free Subgroups of Groups Acting on Trees with Inversions

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Abstract

A group is termed quasi-free if it is a free product of a free group and of cyclic groups of order 2 and a graph is termed quasi -graph if an edge of the graph equals its inverse is allowed. In this paper we show that the fundamental group of the connected quasi-graph is a quasi-free group and we show that if G is a group acting on a tree X with inversions then the quotient graph X/G is a connected quasi-graph and the fundamental group $\pi(X/G)$ is a quasi-free group. Furthermore, we find the structures of quasi-free subgroups of G.

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1 Introduction

In [7], H. Bass, and, J-P Serre, obtained the structures of groups acting on trees under the conditions that the edge of a graph does not equal its inverse, and the actions are without inversions. In [6], Mahmud generalized these concepts to the case where the an edge of the graph equals its inverse is allowed, and the actions of groups on trees are with inversions. The aim of this paper is to generalize the results related free subgroups of groups acting on trees without inversions to the case where the actions of groups on trees are with inversions. This paper is divided into 5 sections. In section 2, we introduce the concepts of a quasi -graph. In section 3 we show that if X is a connected quasi-graph, then the fundamental group π(X) of X is a quasi-free group. In section 4, we introduce the concepts of the actions of groups on graphs with inversions, fundamental domains, and quotient graphs, and show that if G is a group acting on the tree X with inversions, and if X/G the quotient graph induced by the action of G on X, then the fundamental group π(X/G) of X/G is quasi-free group. Furthermore, we find the structures of groups acting on trees with inversions and obtain some results related to quasi-free subgroups. In section 5 we find the structures of quasi-free subgroups of groups acting on trees with inversions.

2 Preliminaries

In this section we recall some definitions which will be used in the sequel.

A quasi-graph X consists of two disjoint sets V(X),(the vertices of X) and E(X), (the edges of X), with V(X) non-empty, together with three functions ∂_0: E(X)$\rightarrow$ V(X),∂_1: E(X)$\rightarrow$V(X), and η:E(X)$\rightarrow$E(X) satisfying the conditions that $\partial_0\eta=\partial_1, \partial_1\eta=\partial_0$, and η is an involution fixing some elements of E(X). For simplicity, if e$\in$E(X), we write ∂_0(e) = o(e), ∂_1(e) = t(e), and η(e) = $\bar{e}$.

This implies that o($\bar{e}$) = t(e), t($\bar{e}$) = o(e), and $\bar{\bar{e}}$ = e. If $\bar{e}$ = e, then e is called a quasi-edge.

For example, if X is a non-empty finite set such that X and X$\times$X are disjoint then Γ = X$\cup$(X$\times$X) forms a quasi-graph where the vertex set V(Γ) = X, and the edge set E(Γ) = X$\times$X, and for the edge (x, y) we have o(x, y) = x, t(x, y) = y, and $\overline{(x,y)}$ = (y, x). Moreover, for every x$\in$X, (x, x) is a quasi-edge.

An orientation of a graph X is a set of edges of X consisting of exactly one member of each of the set {e, $\bar{e}$} for e$\neq\bar{e}$, together with every edge e such that e = $\bar{e}$.

By a path p of the graph X we mean the sequence $p=(v_0, y_1, v_1, y_2, ..., y_n, v_n)$, where $v_0, v_1, ..., v_n$ are vertices of X, and y_1, y_2,..., y_n are edges of X such that $o(y_1)=v_0$, $t(y_i)=v_i$ for i = 1, …, n, and $t(y_j)=o(y_{j+1})$ for j = 1, …, n-1.

For simplicity we write $p: y_1, y_2, ..., y_n$.

We have the following concepts related to the path p defined above.

[1] If n = 0 we write $p: 1_{v_0}$, and say that p is the *trivial* path of v_0.

[2] n is called the length of p, and is denoted by $|p|$ = n.

[3] The *origin* o(p) and terminal t(p) are defined as o(p) = o(y_1), and t(p) = t(y_n) and say that p is a path in X joining the vertices v_0 and v_n.

We define o(1_v) = v, and t(1_v) = v.

[4] p is called closed if o(p) = t(p).

[5] The inverse of p is denoted by $\bar{p}$ and defined as the sequence of edges

$\bar{y}_n,...,\bar{y}_2,\bar{y}_1$. It is clear that $\bar{\bar{p}}$ = p, and $\bar{p}$ is a path joining v_n and v_0. Moreover, if p is closed, then $\bar{p}$ is closed.

[6] p is called reduced if n = 0 or $y_{i+1} \neq \bar{y}_i$ for 1 $\leq$ i $\leq$ n. It is clear that if P is reduced, then $\bar{p}$ is reduced.

[7] If q is a path of X then we say that q is obtained from p by elementary reduction if for some i, $y_{i+1} = \bar{y}_i$ and q is the path $y_1, y_2,..., y_{i-1}, y_{i+2},..., y_n$.

[8] If q is the path $x_1, x_2,..., x_m$ of X such that t(p) = o(q), then pq is defined to be the path $y_1, y_2,...., y_n, x_1, x_2,..., x_m$.

It is clear that o(pq) = o(p), t(pq) = t(q), and $\overline{pq} = \bar{q}\,\bar{p}$.

A quasi-graph X is connected if for every pair of vertices u, v in X there is a path in X from u to v, and is a tree if it is connected, and for any pair of vertices there is exactly one reduced path joining them.

A graph Y is called a subgraph of the graph X if V(Y) $\subseteq$ V(X) and E(X) $\subseteq$ E(X). If Y is a tree then Y is called a subtree of X.
If Y is a subtree of X such that V(Y) = V(X) then Y is a maximal subtree of X.

3 Fundamental groups of quasi-graphs

In this section we obtain the structures of the fundamental groups of connected quasi-graphs. Let X be a connected quasi-graph, p and q be two paths of X. Then we say that p is homotopi*c* to q, written p $\approx$ q, if there is a sequence of paths p_1 = p, …, p_n = q in X such that for i < n one of p_i, p_{i+1} is an elementary reduction of the other. It is clear that if p $\approx$ q, then o(p) = o(q), t(p) = t(q), and $\bar{p} \approx \bar{q}$.

Let v_0 be a fixed vertex of X. Then it is clear that $\approx$ is an equivalence relation on the set of all closed paths of X at v_0. For the closed path p of X at v_0, define [p] to be the class of all paths in X equivalent to p under $\approx$, and π(X; v_0) be the set of all equivalent classes under $\approx$. That is,
π(X; v_0) = {[p]: p is a path in X, o(p) = t(p) = v_0}.
π(X; v_0) forms a group under multiplication [p][q] by [p][q] = [pq], where $[1_{v_0}]$ is the identity element in π(X; v_0) and the inverse of the class [p] is the class $[\bar{p}]$, i.e., $[p]^{-1} = [\bar{p}]$. π(X; v_0) is called the fundamental group of X with respect to v_0. In [3] it is shown that if $u_0 \in$ V(X), then π(X; v_0) and π(X; u_0) are isomorphic. So we speak of the fundamental group of X and denote it by π(X).

Now we show that the fundamental group of a connected quasi-graph is a quasi-

free group, where a group is termed quasi free- group if it is a free product of copies of C_∞ and C_2 where C_∞ denotes infinite cyclic group, and C_2 a cyclic group of order 2. The following are examples of quasi-free groups.
(1) Every free group is a quasi-free group. That is, a free product of copies of C_∞ and a zero number of copies of C_2;
(2) The group $C_\infty * C_\infty * C_2$ is a quasi-free group;
(3) The infinite dihedral group $C_2 * C_2$ is a quasi-free group.
For the rest of this section let X be a connected quasi-graph, v_0 be a fixed vertex of X, T be a maximal subtree of X, and A be an orientation of the graph X– E(T). Then $v_0 \in$V(T) and for each u$\in$V(X), there exists a unique reduced path denoted p_u in T joining v_0 and u with convention that p_{v_0} is the trivial path. For each edge y$\in$E(X), let p_y be the closed path $p_y = p_{o(y)}\, y\, \overline{p}_{t(y)}$.
It is clear that o(p_y) = t(p_y) = v_0, and $p_{\bar{y}} = \overline{p}_y$.
Now we give a summary of the proof of the following lemma. For details we refer the readers to [3].

Lemma 3.1. π(X; v_0) is generated by [y] and [x], and π(X; v_0) has the presentation $\langle y, x \mid x^2 = 1\rangle$ via the mapping y→[y], x→[x] where y, x$\in$A, $\bar{y} \neq y$, and $\bar{x} = x$.

Proof. Let p be the path $y_1, y_2, ..., y_n$ such that o(p) = t(p) = v_0. Then it is clear that [p] = $[p_{y_1}][p_{y_2}].....[p_{y_n}]$. Consequently π(X; v_0) is generated by [e], for all e$\in$E(X). Let G be the group of the presentation
$\langle E \mid R\rangle = \langle e, y, x \mid e=1, \bar{y} = y^{-1}, x^2 = 1\rangle$, where e, y, and x are edges of X such that e$\in$E(T), $\bar{y} \neq y$, and $\bar{x} = x$. Then by Tietze transformations [2, p. 50] we delete the edges e and $\bar{y}$, and the relations e = 1, $\bar{y} = y$ from the generating symbols and the relations of the presentation of G yields that G has the presentation $\langle y, x \mid x^2 = 1\rangle$, where y, x$\in$A, $\bar{y} \neq y$, and $\bar{x} = x$. Then G is a free product of a free group of base the edges y$\in$E(X) and of cyclic groups $\langle x \mid x^2 = 1\rangle$ of order 2 generated by x$\in$A. Thus, G is a quasi-free group. Now we show that π(X; v_0) is isomorphic to G. Let F be a free group of base E and H be the normal subgroup of F generated by the relations of R so that G = F/H. Let $\lambda : E \to \pi$(X; v_0) be the function given by $\lambda(z) = [p_z]$, z$\in$E, where p_z is as above. Then it is clear that $\lambda(E)$= {[p_z] : z$\in$E(X)}. Then λ induces a unique epimomorphism $\tilde{\lambda}$: F→ π(X; v_0) given by $\tilde{\lambda}\,(e_1 e_2 ... e_n) = \lambda(e_1)\lambda(e_2)...\lambda(e_n)$,

where w = $e_1, e_2, ..., e_n$ is a reduced word of F. Let H = ker($\tilde{\lambda}$). For the edge e of T it is clear that $p_{o(e)}e = p_{t(e)}$ or $p_{o(e)} = p_{t(e)}\bar{e}$. This implies that $p_e \approx 1_{v_0}$. Then $\lambda(e) = [p_e] = [1_{v_0}]$ and the relation $e = 1$ holds in π(X; v_0).
For the edge y not in T, $\bar{y} \neq y$ we have $p_y p_{\bar{y}} \approx 1_{v_0}$ Then $\lambda(y)\lambda(\bar{y}) = [1_{v_0}]$ and the relation $\bar{y} = y^{-1}$ holds in π(X; v_0). For the edge x not in T, $\bar{x} = x$, we have $\bar{p}$ = p. This implies that $p_x p_{\bar{x}} = p_x^2 \approx 1_{v_0}$. Then $(\lambda(x))^2$ = $[1_{v_0}]$ and the relation $x^2 = 1$ holds in π(X; v_0). This implies that H $\leq$ ker($\tilde{\lambda}$). Now we show that ker($\tilde{\lambda}$) $\leq$ H. Let w = $e_1, e_2, ..., e_n$ be a reduced word of F such that w $\notin$ H. Then $e_{i+1} \neq e_i^{-1}$ for i = 1, …, n-1, and there exists at least one edge $e_i \in \{e_1, e_2, ..., e_n\}$ such that e_i is not in T, and $e_{i+1} \neq \bar{e}_i$. We need to show that $\tilde{\lambda}(w) \neq [1_{v_0}]$. Let p be the path $p_{e_1} p_{e_2} \dots p_{e_n}$. By deleting the edges e_i and e_{i+1} from p if $e_{i+1} = \bar{e}_i$ from p yields a reduced path q. This implies that $\tilde{\lambda}(w) = \tilde{\lambda}(p) = \tilde{\lambda}(q) \neq [1_{v_0}]$. Hence H = ker($\tilde{\lambda}$) and G $\cong \pi$(X; v_0).
This completes the proof.

In [3] it was shown that if u and v are two vertices of the connected quasi-graph X then π(X; u) $\cong$ π(X; v) and the presentation type of such groups are independent of all choices of maximal subtrees and orientations of X. So we agree to speak of the fundamental group of X and write π(X).
We conclude this section the following theory.

Theorem 3.2. The fundamental group π(X) of the connected quasi-graph X is a quasi-free group of rank $|A|$ for any maximal subtree T of X and any orientation A of the graph X – E(T). Moreover, if X is a tree then π(X) is trivial.

4 Fundamental groups of quotient-graphs

In this section we show that groups acting on trees with inversions induce connected quasi-graphs of quasi-free groups as their fundamental groups. In symbols, we show if G is a group acting on the tree X with inversions then the fundamental group $\pi(X/G)$ of the quotient graph X/G is a quasi-free group. Moreover, we show that if the stabilizers of X under G are trivial, then G is a quasi-free group and G $\cong$ $\pi(X/G)$. We start the following concepts.
If X and Y are two graphs, then the mapping f: X$\to$Y is called a morphism if f takes the vertices of X to the vertices of Y, and the edges of X to the edges of Y such that if e$\in$E(X), then o(f(e)) = f(o(e)), t(f(e)) = f(t(e)), and $\overline{f(e)} = f(\bar{e})$.

If G is a group, and X is a non-empty graph, then it is easy to show that the set G×X forms a graph, where V(G×X) = G×V(X), E(G×X) = G×E(X), and if g∈G, and y∈E(X), then o(g, y) = (g, o(y)), t(g, y) = (g, t(y)), and $\overline{(g,y)}=(g,\bar{y})$.
We say that a group G acts on a graph X if there is a graph morphism from G×X to X, or equivalently, if there is a group homomorphism
ϕ: G $\rightarrow$ Aut(X), where Aut(X) is the set of all automorphisms of the graph X which is a group under the composition of morphisms. In this case, if x∈X (vertex or edge) and g∈G, we write g(x) for (ϕ(g))(x). Thus, if g∈G, and y∈E(X), then g(o(y)) = o(g(y)), g(t(y)) = t(g(y)), and g($\bar{y}$) = $\overline{g(y)}$.
The case g(y) = $\bar{y}$ for g∈G and y∈E(X) may occur. That is; G acts with inversions on X. For more details of groups acting on graphs without inversions we refer the readers to [1] and [7], and with inversions to [5] and [6].

The following notations are needed for the rest of the paper.
Let G be a groups acting on a connected quasi-graph X with inversions.
(1) If x∈X, define G(x) to be the set G(x) = {g(x): g∈G}. This set is called the orbit of x. It is clear that G(x) ⊆ X, and G(x) = G(y) if and only if x and y are in the same orbit.
(2) For any Y⊆ X let G(Y) = {G(y): y∈Y}, and let
G(X) = X/G = { G(z): z∈X},
(3) If x, y∈X, define G(x→y) to be the set G(x→y) ={g∈G: g(x) = y}, and G(x→x) = G_x, the stabilizer of x. It is clear that if y∈E(X) and u∈{o(y), t(y)}, then $G_{\bar{y}} = G_y$ and $G_y \le G_u$.

Definition 4.1. Let G be a group acting on a tree X with inversions, T and Y be two subtrees of X such that T is contained in Y, and each edge of Y has at least one end in T, and T and Y satisfying the following.
(i) T contains exactly one vertex from each vertex orbit.
(ii) Y contains exactly one edge y(say) from edge orbit such that G(y→$\bar{y}$) = Ø, and exactly one pair x, $\bar{x}$ from each edge orbit such that G(x→$\bar{x}$) ≠ Ø. Then the pair (Y; T) is called a fundamental domain for the action of G on X.

Remark. For the rest of this paper let G, X, T, and Y be as in Definition 4.1.
Definition 4.2. Let V, E, E_0, E_1 and E_2 be the sets defined as follows.
(1) V = V(T), the vertices of T;
(2) E = E(Y), the edges of Y;
(3) E_0 = E(T), the edges of T,
(4) E_1 = {e∈E: o(e)∈V, t(e)∉V, G(e→$\bar{e}$) = Ø}, and
(5) E_2 = {e∈E: o(e)∈V, t(e)∉V, G(e→$\bar{e}$) ≠ Ø}.

It is clear that $E = E_0 \cup E_1 \cup E_2 \cup \overline{E_1} \cup \overline{E_2}$, where $\overline{E_i} = \{\overline{e} : e \in E_i\}, i = 1, 2$ and $E_1 \cup E_2$ is an orientation of the subgraph Y – E(T) of X.
The proofs of the following propositions are clear.

Proposition 4.3. Let π (T; Y) be the group of the presentation π (T; Y) = $\langle y, x \mid x^2 = 1 \rangle$, where y$\in E_1$ and y$\in E_2$. Then π (T; Y) is a quasi-free group of rank $|E_1 \cup E_2|$.

Proposition 4.4. (i) For any v∈V(X), there exists a unique vertex denoted v* ∈V such that G(v*→ v) ≠ Ø. That is, g(v*) = v, g∈G.
v* is called the representative of v under G.
(ii) For any e∈E(X), there exists e*∈E such that G(e*→ e) ≠ Ø.
It is clear that e*∈ E_1 if G(e→$\overline{e}$) = Ø and e*∈ E_2 if G(e→$\overline{e}$) ≠ Ø.

Lemma 4.5. G(X) = X⁄G forms a connected quasi-graph and π(X⁄G) is a quasi-free group such that π(X⁄G) ≅ π (T; Y). If Y = T then X⁄G is a tree.

Proof. Let V(X⁄G) = {G(v): v∈V(X)}, and E(X⁄G) = {G(e): e∈E(X)}. The action of G on X implies that for any element g of G, any vertex v and any edge e of X, we have g(e) ≠ v. This implies that G(v)∩G(e) = ∅. Consequently, V(X⁄G) ∩E(X⁄G) = ∅. For the edge e∈E(X), let o(G(e)) = G(o(e)), t(G(e)) = G(t(e)), and $\overline{G(e)} = G(\overline{e})$.
This implies that X⁄G forms a graph. The action of G on X with inversions implies that there exist an element g of G, and an edge e of X such that g(e) = $\overline{e}$. This implies that G(g(e)) = G($\overline{e}$). Then G(e) = $\overline{G(e)}$. Consequently, X⁄G forms a quasi-graph. It is clear that the structure of X⁄G implies that the mapping p: X→ X⁄G given by p(x) = G(x) is a surjective morphism. Then p(X) = X⁄G. Since X is connected, therefore X⁄G is connected and by Theorem 3.2, π(X⁄G) is a quasi-free group. It is clear that Definitions 4.1, 4.2, and Proposition 4.4 imply that G(X) = X⁄G, G(T) is a maximal subtree of X⁄G and G($E_1 \cup E_2$) is an orientation of the subgraph G(X) – E(G(T)). For each y∈ E_1 let $\widetilde{y}$ = G(Y) and for each x∈ E_2 let $\widetilde{x}$ = G(X). Then by Theorem 3.2 π(X⁄G) is a quasi-free group such that π(X⁄G) ≅ $\langle \widetilde{y}, \widetilde{x} \mid \widetilde{x}^2 = 1 \rangle$. Then by Proposition 4.3 we have π(X⁄G) ≅ π (T; Y). If Y = T then G(Y) = G(X) = X⁄G = G(T) on which X⁄G is a tree. This completes the proof.

Remark. X⁄G is called the quotient graph for the action of G on X.

Definition 4.6. For every e∈E let [e] be an element of G be chosen such that t(e) = [e]((t(e))*), and [e] satisfies the following.

(1) If e$\in E_0$ then [e] = 1;

(2) If e$\in E_1$, then $[\bar{e}]=[e]^{-1}$;

(3) If e$\in E_2$, then $[\bar{e}]=[e]$.

Definition 4.7. For y$\in$E let –y be the edge –y = $[y]^{-1}$(y) if o(y)$\in$V and –y = y if t(y)$\in$V and let +y be the edge +y = [y](-y).
It is clear that +y = y if o(y)$\in$V and +y = [y](y) if t(y)$\in$V. Moreover, t(-y)=(t(y))* , o(+y) = (o(y))*, $G_{-y} \leq G_{(t(y))^*}$, and $G_{+y} \leq G_{(o(y))^*}$.
If y$\in E_0 \cup E_2$ then $G_{-y} = G_{+y} = G_y$.

Convention. The following notations are needed to find the structures of groups acting on trees with inversions.

(1) If v$\in$V let$\langle gen(G_v) | rel(G_v) \rangle$ stand for any presentation of G_v, where $gen(G_v)$ is the set of generating symbols of G_v and $rel(G_v)$ is the set of relations of G_v.

(2) If m$\in E_0$ let $G_m = G_m$ stand for the set of relations $w(g) = w'(g)$, where g is in the set of generators of G_m, and $w(g)$ and $w'(g)$ are words in the generating symbols of $G_{t(m)}$ and $G_{o(m)}$ of value g.

(3) If y$\in E_1$ let $y.[y]^{-1}G_y[y].y^{-1} = G_y$ stand for the set of the relations $y.w(g).y^{-1} = w([y]g[y]^{-1})$, where g is in the set of generating symbols of G_y, and, $w(g)$ and $w([y]^{-1}g[y])$ are words in the generating symbols of $G_{(t(y))^*}$ and $G_{o(y)}$ of values g and $[y]g[y]^{-1}$ respectively.

(4) If x$\in E_2$ let $x.G_x.x^{-1} = G_x$ stand for the set of the relations $x.w(g).x^{-1} = w([x]g[x]^{-1})$, where g is in the generating symbols of G_x, and, $w(g)$ and $w([x]g[x]^{-1})$ are words in the generating symbols of $G_{o(x)}$ of values g and $[x]g[x]^{-1}$ respectively. Moreover, $x^2 = [x]^2$ stands for the relation $x^2 = w([x]^2)$, where $w([x]^2]$ is a word in the generating symbols of $G_{o(x)}$ of value$[x]^2$.

Proposition 4.8. G is generated by [y], [x] and the generators ofG_v, where y$\in E_1$, x$\in E_2$ and v$\in$V, and G has the presentation$\langle S(Y) | R(Y) \rangle$, where S(Y) is the set of generating symbols of the following forms.

(1) gen(G_v), the set of generating symbols of G_v, where v$\in$V;

(2) The set of symbols y, where y$\in E_1$;

(3) The set of symbols x, where x$\in E_2$.
R(Y) is the set of relations the following forms
(1) rel(G_v), the set of relations of the presentation of G_v, v$\in$V;
(2) $G_m = G_m$, m$\in E_0$;
(3) $y.[y]^{-1}G_y[y].y^{-1} = G_y$, y$\in E_1$;
(4) $x.G_x.x^{-1} = G_x$; x$\in E_2$;
(5) $x^2 = [x]^2$, x$\in E_2$.
Equivalently, G has the presentation
$\left\langle gen(G_v), y, x \middle| rel(G_v), G_m = G_m, y.[y]^{-1}G_y[y].y^{-1} = G_y,\ x.G_x.x^{-1} = G_x, x^2 = [x]^2 \right\rangle$
via the mapping $G_v \to G_v$, y→[y], x→[x].

Proof. See [6, Th. 5.1].

We have the following corollaries of Proposition 4.8.

Corollary 4.9. If G acts on X without inversions then G has the presentation $\left\langle gen(G_v), y \middle| rel(G_v), G_m = G_m, y.[y]^{-1}G_y[y].y^{-1} = G_y \right\rangle$ via the mapping $G_v \to G_v$, y→[y], where v$\in$V and y$\in E_1$.

Proof. Since G acts on X without inversions therefore $E_2 = \varnothing$ and Proposition 4.8 implies the required presentation.

Corollary 4.10. If T = Y then G is the tree product $\left\langle gen(G_v) \middle| rel(G_v), G_m = G_m \right\rangle$ of the groups G_v with amalgamation subgroups G_m and X⁄G is a tree where v$\in$V and m$\in E_0$.

Proof. The condition T = Y implies that $E_1 \cup E_2 = \varnothing$ and G is the tree product of the subgroups G_v with amalgamation subgroups G_m, and π(T; Y) is trivial. Moreover, Lemma 4.5 implies that X⁄G is a tree.

Corollary 4.11. If $G_v = \{1\}$ for all v$\in$V(X), then G is a quasi-free group such that G ≅ π(X⁄G).

Proof. By substituting $G_v = \{1\}$, $G_m = \{1\}, G_y = \{1\}$ and $G_x = \{1\}$ for all v$\in$V m$\in E_0$, y$\in E_1$ and x$\in E_2$ in the presentation of G in Proposition 4.8 yields G has the presentation G = $\left\langle y, x \middle| x^2 = [x]^2 \right\rangle$ = π(T; Y). Then G is a quasi-free group and Lemma 4.5 implies that G ≅ π(X⁄G).

The main result of this section is the following theorem which is a generalization of Proposition 4.4 of [1, page 17] and of different proof.

Theorem 4.12. Let H be the subgroup of G generated by G_v, v∈V(X). Then
[1] H is a normal subgroup of G;
[2] X/H is a tree,
[3] G/H acts on X/H with inversions,
[4] G/H is a quasi-free group and $G/H \cong \pi(X/G)$.

Proof. **[1]** For g∈G , v∈V(X) and h∈H we have $gG_vg^{-1} = G_{g(v)}$ and the generators of H imply that $h = h_1h_2 \ldots h_n$ where $h_i \in G_{v_i}$ and $v_i \in V(X)$ for i = 1, 2, .., n. Then $ghg^{-1} = gh_1g^{-1}gh_2g^{-1} \ldots g^{-1}gh_ng^{-1}$ and $gh_ig^{-1} \in gG_{v_i}g^{-1} = G_{g(v_i)} \le H$ for i = 1, 2, .., n. Consequently H is a normal subgroup of G.
[2] For v∈V(X) we have $H_v = H \cap G_v = G_v$ because $G_v \le H$. The action of H on X implies that H is generated by the stabilizers of the vertices of any tree of representatives for the action of H on X. Then Corollary 4.10 implies that $\pi(X/H)$ is trivial, or equivalently X/H is a tree.
[3] For g∈G and x∈X (vertex or edge), define gH(H(x)) = H(g(x)). It is clear that G/H takes vertices to vertices and edges to edges of X/H.
Furthermore, o(gH(H(e))) = gH(o(H(e)))), t(gH(H(e))) = gH(t(H(e)))), and $\overline{gH(H(e))} = gH(\overline{H(e)}) = gH(H(\overline{e}))$. The action of G on X with inversions implies that there exist g∈G and e∈E(X) such that g(e) = $\overline{e}$. Then $H(g(e)) = gH(H(e)) = H(\overline{e}) = \overline{H(e)}$. Hence G/H acts on X/H with inversions.
[4] Let x∈X and g∈G. Then

$$\begin{aligned} gH(H(x)) = H(x) &\Rightarrow H(g(x)) = H(x) \\ &\Rightarrow gh(x) = x,\ h \in H \\ &\Rightarrow gh \in G_x \\ &\Rightarrow g \in G_x h^{-1} \\ &\Rightarrow g \in G_x H \\ &\Rightarrow gH \in G_xH/H \\ &\Rightarrow (G/H)_{H(x)} = G_xH/H = H/H \text{ because } G_x \le H. \end{aligned}$$

This implies that the stabilizer of gH(x) is H/H, the identity element of G/H. Then Corollary 4.11 implies that G/H is a quasi-free group and $G/H \cong \pi(\frac{X/H}{G/H}) \cong \pi(X/G))$. This completes the proof.

Corollary 4.13. If G acts without inversions on X then H is a normal, X/H is a tree, G/H acts on X/H without inversions and G/H is a free group.

5 Quasi-Free Subgroups

In this section we use Theorem 4.7 of [4] to find the structures of the quasi-free subgroups of groups acting on trees with inversions. Let G be a group acting on a tree X with inversions, (T; Y) be a fundamental domain for the action of G on X, V, E, E_0, E_1, and E_2 be the sets of edges introduced in Definition 4.2.

Let H be a subgroup of G such that H$\cap G_v$= {1}for all v$\in$V(X). The purpose of this section is to show that H is a quasi-free subgroup of G and then find the generators of H. In view of Definition 4.2 of [4], if v$\in$V and e$\in E_0 \cup E_1 \cup E_2$ we have the following.

(a) D_e and $D_{\bar{e}}$ are any left coset representative systems for $G_{o(e)}$ modG_e and $G_{(t(e))^*}$ mod G_{-e} respectively containing 1, but otherwise arbitrary.

(b) D_v is a double coset representative system for G mod (H,G_v) satisfying the condition that if g$\in D_v$, g $\neq$ 1, g = $g_0[y_1]g_1[y_2]g_2..[y_n]g_n$, where n ≥ 0, $y_i \in$E, for i= 1, 2, … , n such that $g_0 \in G_{(o(y_1))^*}$, $g_i \in G_{(t(y_i))^*}$, for i = 1, 2 , … , n, $(t(y_i))^* = (o(y_{i+1}))^*$, for i= 1, 2, … , n-1, and $(t(y_n))^*$ = v, then $f_i \in D_{(o(y_{i+1}))^*}$, and $D_{y_{i+1}}$ where $f_i = g_0[y_1]g_1[y_2]g_2..[y_i]$ for i = 1, 2, ..., n-1.

Lemma 4.3 of [4] implies that for any e$\in E_0 \cup E_1 \cup E_2$ and g$\in$G then there exist unique elements denoted $\overline{g[e]} \in D_{(t(e))^*}$, $\overline{\overline{g[e]}} \in D_{\bar{e}}$, and $g_e \in G_e$ such that

$$\delta_{g,e} = gg_e[e]\overline{\overline{g[e]}}^{-1}\overline{g[e]}^{-1} \in H.$$

It is clear that if g[e]$\in D_{(t(e))^*}$, then $\delta_{g,e}$= 1 and if e$\in E_2$, a$\in D_{o(e)}$, b$\in D_e$ such that $H \cap ab[e]G_e b^{-1}a^{-1} \neq \varnothing$, then $\delta_{ab,e} = ab[e]b^{-1}a^{-1}$.

The main result of this section is the following theorem.

Theorem 5.1. Let G be a group acting on a tree X with inversions, (T; Y) be a fundamental domain for the action of G on X., and H be a subgroup of G such that that H$\cap G_v$= {1}for all v$\in$V(X). Then H is a quasi-free subgroup of G generated by $\delta_{ab,m}$, $\delta_{ab,y}$, $\delta_{ab,x}$, and $\delta_{ab,z}$, and H has the presentation

H = $\left\langle ab(m),\ ab(y),\ ab(x),\ ab(z)\ \middle|\ (ab(z))^2 = 1 \right\rangle$ via the mapping

ab(m) $\to$ $\delta_{ab,m}$, m$\in E_0$, a$\in D_{o(m)}$ and b$\in D_m$ such that ab$\notin D_{t(m)}$;
ab(y) $\to$ $\delta_{ab,y}$, y$\in E_1$, a$\in D_{o(y)}$ and b$\in D_y$ such that ab[y]$\notin D_{(t(y))*}$;
ab(x) $\to$ $\delta_{ab,x}$, x$\in E_2$, a$\in D_{o(x)}$ and b$\in D_x$ such that ab[x]$\notin D_{o(x)}$ and $H \cap ab[x]G_x b^{-1}a^{-1} = \varnothing$;
ab(z) $\to$ $\delta_{ab,z}$, z$\in E_2$, a$\in D_{o(z)}$ and b$\in D_z$ such that ab[z]$\notin D_{o(z)}$ and $H \cap ab[z]G_z b^{-1}a^{-1} \neq \varnothing$.

Proof. H acts on X and for each p$\in$X (vertex or edge) we have $H_p = H \cap G_p = \{1\}$. Then Corollary 4.12 yields that H is a quasi-free subgroup of G. In view of Theorem 4.7 of [4], the condition
H$\cap aG_v a^{-1}$ = {1}, a$\in D_v$, v$\in$V implies that H has the set of generators of the given forms.
The conditions H$\cap abG_m b^{-1}a^{-1}$ = H$\cap abG_m b^{-1}a^{-1}$ = {1},
H$\cap abG_{-y}b^{-1}a^{-1}$ = $H \cap \delta_{ab,y}G_{-y}\delta^{-1}_{ab,y}$ = {1},
H$\cap abG_x b^{-1}a^{-1}$ = $H \cap \delta_{ab,x}G_x\delta^{-1}_{ab,x}$ = {1}, and
H$\cap abG_z b^{-1}a^{-1}$ = $H \cap \delta_{ab,z}G_z\delta^{-1}_{ab,z}$ = {1},
imply that H has the given set of generating symbols .
The conditions and $H \cap ab[x]G_x b^{-1}a^{-1} = \varnothing$ and $H \cap ab[z]G_z b^{-1}a^{-1} \neq \varnothing$ imply that H has the relations $(ab(z))^2 = 1$. This completes the proof.
By taking the set of edges $E_2 = \varnothing$ yields the following corollary.

Corollary 5.2. Let G be a group acting on a tree X without inversions, (T; Y) be a fundamental domain and H be a subgroup of G such that that H$\cap G_v$ = {1}for all v$\in$V(X). Then H is a free subgroup of G generated by the elements $\delta_{ab,m}$, m$\in E_0$, a$\in D_{o(m)}$ and b$\in D_m$ such that ab$\notin D_{t(m)}$ and by the elements $\delta_{ab,y}$, y$\in E_1$, a$\in D_{o(y)}$ and b$\in D_y$ such that ab[y]$\notin D_{(t(y))*}$.

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