

## On Edge-Balance Index Sets of Wheels

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### Abstract

Let  $G$  be a simple graph with vertex set  $V(G)$  and edge set  $E(G)$ , and let  $\mathbb{Z}_2 = \{0, 1\}$ . Any edge labeling  $f$  induces a partial vertex labeling  $f^+ : V(G) \rightarrow \mathbb{Z}_2$  assigning 0 or 1 to  $f^+(v)$ ,  $v$  being an element of  $V(G)$ , depending on whether there are more 0-edges or 1-edges incident with  $v$ , and no label is given to  $f^+(v)$  otherwise. For each  $i \in \mathbb{Z}_2$ , let  $v_f(i) = |\{v \in V(G) : f^+(v) = i\}|$  and let  $e_f(i) = |\{e \in E(G) : f(e) = i\}|$ . An edge-labeling  $f$  of  $G$  is said to be edge friendly if  $|\{e_f(0) - e_f(1)\}| \leq 1$ . The edge-balance index set of the graph  $G$  is defined as  $EBI(G) = \{|v_f(0) - v_f(1)| : f \text{ is edge-friendly}\}$ . In this paper, we investigate and present results concerning the edge-balance index sets of wheels.

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**Keywords:** vertex labeling, edge labeling, friendly labeling, cordiality, edge-balance index set, wheel.

# 1 Introduction

In [3], Kong and second author considered a new labeling problem of graph theory. Let  $G$  be a simple graph with vertex set  $V(G)$  and edge set  $E(G)$ , and let  $\mathbb{Z}_2 = \{0, 1\}$ . An edge labeling  $f : E(G) \rightarrow \mathbb{Z}_2$  induces a vertex partial labeling  $f^+ : V(G) \rightarrow \mathbb{Z}_2$  defined by  $f^+(v) = 0$  if the edges labeled 0 incident on  $v$  is more than the number of edges labeled 1 incident on  $v$ , and  $f^+(v) = 1$  if the edges labeled 1 incident on  $v$  is more than the number of edges labeled 0 incident on  $v$ .  $f^+(v)$  is not defined if the number of edges labeled by 0 is equal to the number of edges labeled 1. For  $i \in \mathbb{Z}_2$ , let  $v_f(i) = |\{v \in V(G) : f^+(v) = i\}|$ , and let  $e_f(i) = |\{e \in E(G) : f(e) = i\}|$ .

With these notations, we now introduce the notion of an edge-balanced graph.

**Definition 1.1** An edge labeling  $f$  of a graph  $G$  is said to be *edge-friendly* if  $|e_f(0) - e_f(1)| \leq 1$ . A graph  $G$  is said to be an *edge-balanced* graph if there is an edge-friendly labeling  $f$  of  $G$  satisfying  $|v_f(0) - v_f(1)| \leq 1$ .

Chen, Lee, et al in [1] proved that all connected simple graphs except the star  $K_{1,2k+1}$ , where  $k \geq 0$  are edge-balanced.

**Definition 1.2** The *edge-balance index set* of the graph  $G$ ,  $\text{EBI}(G)$ , is defined as  $\{|v_f(0) - v_f(1)| : \text{the edge labeling } f \text{ is edge-friendly}\}$ .

We will use  $v(0)$ ,  $v(1)$ ,  $e(0)$ ,  $e(1)$  instead of  $v_f(0)$ ,  $v_f(1)$ ,  $e_f(0)$ ,  $e_f(1)$ , provided there is no ambiguity.

**Example 1.3**  $\text{EBI}(nK_2)$  is  $\{0\}$  if  $n$  is even and  $\{2\}$  if  $n$  is odd.

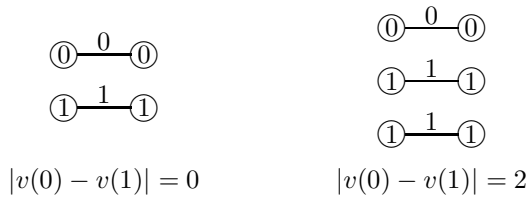


Figure 1: The edge-balance index set of  $2K_2$  and  $3K_2$

For any  $n \geq 1$ , we denote the tree with  $n + 1$  vertices of diameter two by  $\text{St}(n)$ . The star has a center  $c$  and  $n$  appended edges from  $c$ .

**Example 1.4** The edge-balance index set of the star  $\text{St}(n)$  is

$$\text{EBI}(\text{St}(n)) = \begin{cases} \{0\} & \text{if } n \text{ is even,} \\ \{2\} & \text{if } n \text{ is odd.} \end{cases}$$

**Example 1.5** In [10], Lee, Lo and Tao showed that

$$\text{EBI}(P_n) = \begin{cases} \{2\} & \text{if } n \text{ is } 2, \\ \{0\} & \text{if } n \text{ is } 3, \\ \{1, 2\} & \text{if } n \text{ is } 4. \\ \{0, 1\} & \text{if } n \text{ is odd and greater than } 3, \\ \{0, 1, 2\} & \text{if } n \text{ is even and greater than } 4. \end{cases}$$

Figure 2 shows the EBI of  $P_3$  and  $P_4$ .

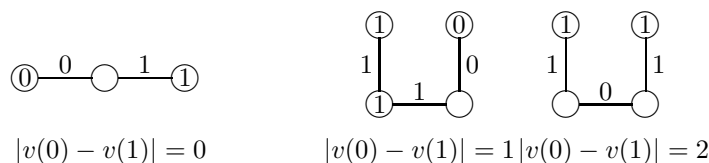


Figure 2: The edge-balance index set of  $P_3$  and  $P_4$

**Example 1.6** Figure 3 shows that the edge-balance index set of a tree with six vertices is  $\{0, 1, 2\}$ .

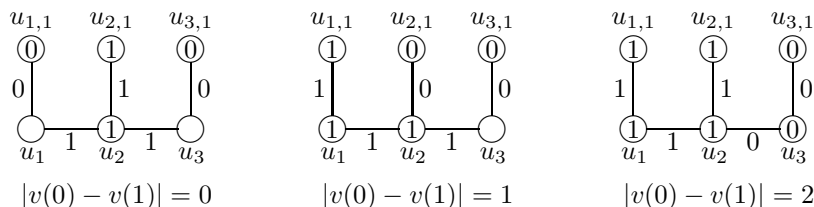


Figure 3: The edge-balance index set of a tree with six vertices

The edge-balance index sets can be viewed as the dual of balance index sets. The balance index sets of graphs were considered in [2, 4, 6, 7, 8, 9, 11, 13]. Let  $G$  be a simple graph with vertex set  $V(G)$  and edge set  $E(G)$ , and let  $\mathbb{Z}_2 = \{0, 1\}$ . A labeling  $f : V(G) \rightarrow \mathbb{Z}_2$  induces an edge partial labeling  $f^* : E(G) \rightarrow A$  defined by  $f^*(vw) = f(v)$ , if and only if  $f(v) = f(w)$  for each edge  $vw \in E(G)$ . For  $i \in \mathbb{Z}_2$ , let  $v_f(i) = \text{card}\{v \in V(G) : f(v) = i\}$  and  $e_{f^*}(i) = \text{card}\{e \in E(G) : f^*(e) = i\}$ . A labeling  $f$  of a graph  $G$  is said to be **friendly** if  $|v_f(0) - v_f(1)| \leq 1$ . If  $|e_f(0) - e_f(1)| \leq 1$  then  $G$  is said to be **balanced**. The **balance index set** of the graph  $G$ ,  $\text{BI}(G)$ , is defined as  $\{|e_f(0) - e_f(1)| : \text{the vertex labeling } f \text{ is friendly}\}$ .

Edge-balance index sets of trees, flower graphs and  $(p, p + 1)$ -graphs were considered in [5, 10, 12]. In this paper, exact values of the edge-balance index sets of wheels are presented.

## 2 On Edge-balance Index Sets of a Cycle

For later use, we provide here some results on the edge-balance index sets of cycles.

**Notation 2.1** Let  $C_n$  be a cycle with a vertex set  $\{c_1, c_2, \dots, c_n\}$ . Let  $f$  be an edge labeling on  $C_n$  (not necessarily edge-friendly). We denote the numbers of edges labeled 0 or 1 by  $f$  by  $e_C(0)$  or  $e_C(1)$ , respectively. We also denote the number of vertices labeled 0, 1, or not labeled by  $f^+$  by  $v_C(0)$ ,  $v_C(1)$ , or  $v_C(\times)$ , respectively.

For a vertex of order 2, with an edge labeling (not necessarily edge-friendly), it can only be labeled in one of the following three ways.

1. If both edges are labeled 0, then the vertex is labeled 0.
2. If both edges are labeled 1, then the vertex is labeled 1.
3. If one edge is labeled 0 and another is labeled 1, then the vertex is not labeled.

If we add an edge to a vertex, then there are two cases:

- A If the vertex was already labeled, then the label of the vertex is not changed after adding an edge because at least two edges are labeled by the same number.
- B If the vertex was not labeled then the label of the vertex is the same as the label assigned to the new edge.

For later reference, we call these as Rule A and Rule B.

**Lemma 2.2** *In a cycle  $C_n$  with an edge labeling  $f$  (not necessarily edge-friendly), we have two equations:*

$$2v_C(0) + v_C(\times) = 2e_C(0)$$

and

$$2v_C(1) + v_C(\times) = 2e_C(1).$$

**Proof.** Every vertex labeled 1 has two incident 1-edges and every unlabeled vertex has one incident 1-edge. No other vertex contains any edge labeled 1. Because every edge is counted twice, we have

$$2v_C(1) + v_C(\times) = 2e_C(1).$$

Similarly, we have

$$2v_C(0) + v_C(\times) = 2e_C(0).$$

□

From these two equations, we can see that  $v_C(\times)$  must be even.

**Corollary 2.3** *In a cycle  $C_n$  with an edge labeling  $f$  (not necessarily edge-friendly),  $v_C(\times)$  is even.*

**Theorem 2.4** *The edge-balance index set of a cycle  $C_n$  is*

$$EBI(C_n) = \begin{cases} \{0\} & \text{if } n \text{ is even,} \\ \{1\} & \text{if } n \text{ is odd.} \end{cases}$$

**Proof.** For an edge-friendly labeling  $f$ , we have  $|v_C(0) - v_C(1)| \leq 1$ . By Lemma 2.2, the edge-balance index is  $|e_C(0) - e_C(1)| = |v_C(0) - v_C(1)|$ . This completes the proof. □

**Example 2.5** Figure 4 shows that the edge-balance index sets  $EBI(C_3) = EBI(C_5) = \{1\}$  and  $EBI(C_4) = EBI(C_6) = \{0\}$

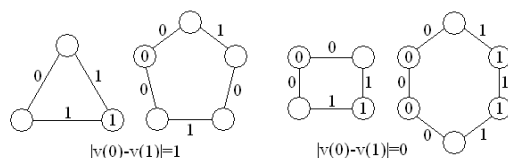


Figure 4: EBI of  $C_3$ ,  $C_4$ ,  $C_5$  and  $C_6$

**Lemma 2.6** *In a cycle  $C_n$  with an edge labeling  $f$  (not necessarily edge-friendly),  $e_C(0)$  or  $e_C(1)$  is zero if and only if  $v_C(\times) = 0$ .*

**Proof.** If  $e_C(0)$  equals zero, then all edges are labeled 1. Thus, all vertices are labeled 1, that is,  $v_C(\times) = 0$ . Similarly, if  $e_C(1)$  equals zero, then  $v_C(\times) = 0$ .

Conversely, if both  $e_C(0) > 0$  and  $e_C(1) > 0$ , then there is at least one 0-edge and one 1-edge in  $C_n$ . Obviously, in  $C_n$ , there must be one 0-edge which meets an 1-edge in a vertex. Thus,  $v_C(\times)$  is not zero. □

**Lemma 2.7** *In a cycle  $C_n$  with an edge labeling  $f$  (not necessarily edge-friendly), if  $v_C(0)$  and  $v_C(1)$  are both positive then  $v_C(\times) > 0$ .*

**Proof.** If both  $v_C(0)$  and  $v_C(1)$  are positive, then we must have at least two edges labeled 0 and two edges labeled 1. Thus, both  $e_C(0)$  and  $e_C(1)$  are greater than one. By Lemma 2.6, we have  $v_C(\times) > 0$ . □

**Lemma 2.8** *In a cycle  $C_n$  with an edge labeling  $f$  (not necessarily edge-friendly), we have*

$$0 \leq v_C(0) \leq e_C(0) - 1$$

and

$$0 \leq v_C(1) \leq e_C(1) - 1.$$

**Proof.** Because we need two 0-edges to get a vertex labeled 0,  $e_C(0)$  0-edges can produce at most  $e_C(0) - 1$  0-vertices. Thus,

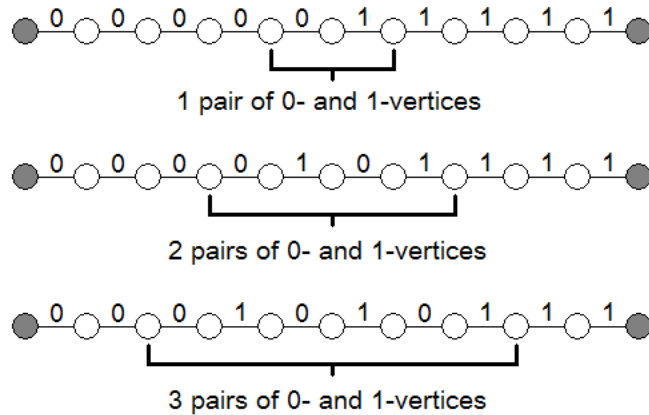
$$0 \leq v_C(0) \leq e_C(0) - 1.$$

Similarly,

$$0 \leq v_C(1) \leq e_C(1) - 1.$$

□

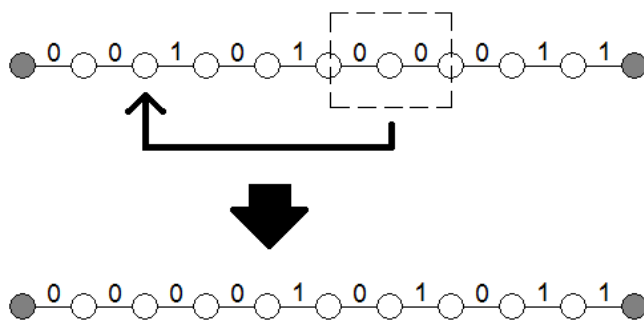
By Corollary 2.3, we know that  $v_C(\times)$  must be even. For an even number  $2k > 0$ , we can construct an edge labeling of  $C_n$  with exactly  $2k$  unlabeled vertices. First, put  $k$  pairs of 0- and 1-edges alternately in the middle of  $P_{n+1}$  and all the remaining 0s on the left side edges and all the remaining 1s on the right side edges. Then, glue the two sided vertices together. (See the following figure.)



Since we put  $k$  pairs of 0 and 1 alternately, these  $2k$  edges produce exactly  $2k - 1$  unlabeled vertices. Additionally, the glued two sided vertices become an unlabeled vertex. All the other vertices are either in the middle of 0-edges or 1-edges. Thus, we have exactly  $2k$  unlabeled vertices.

For an edge labeling  $f$  of  $C_n$  with  $v_C(\times) = 2k > 0$ , we can rearrange it into an edge labeling we just constructed without altering  $v_C(\times)$ . Pick an unlabeled vertex and split this vertex into two vertices. It becomes a  $P_{n+1}$  with the very left edge labeled 0 and the very right edge labeled 1. Let a maximal

chain of 0-edges be a path with only 0-edges where its length is greater than 1 and it is not a subchain of 0-edges with a longer length. If a maximal chain of 0-edges does not contain the very left 0-edge, we can cut the whole chain except one 0-vertex off and insert it into the left side of the first from the left unlabeled vertex without altering the number of 0-vertices, 1-vertices and unlabeled vertices. (See the following figure.)



By repeating this process for all 0-subchains and a similar process for all 1-subchains, it will become the one we constructed previously.

**Lemma 2.9** *In a cycle  $C_n$  with an edge labeling  $f$  (not necessarily edge-friendly), assume that  $v_C(\times) = 2k > 0$ . Then*

$$v_C(1) = e_C(1) - k.$$

**Proof.** Since  $v_C(\times) = 2k > 0$ , by the above rearrangement, we can collect  $k$  pairs of 0- and 1-edges in the middle of  $P_{n+1}$  without altering  $v_C(\times) = 2k$ . Since these  $k$  pairs of edges occupy  $k$  edges labeled 1, there are only  $e_C(1) - k$  1-edges left on the right part of  $P_{n+1}$ . Since the very right edge of the  $k$  pairs of 0- and 1-edges is an edge labeled 1, we have a chain of 1-edges with  $e_C(1) - k + 1$  edges and  $e_C(1) - k$  vertices in between edges. Thus,

$$v_C(1) = e_C(1) - k.$$

□

**Corollary 2.10** *In a cycle  $C_n$  with an edge labeling  $f$  (not necessarily edge-friendly), assume that  $v_C(\times) = 2k > 0$ . Then*

$$v_C(0) = n - e_C(1) - k.$$

**Proof.** By Lemma 2.9, we have

$$v_C(1) = e_C(1) - k.$$

In  $C_n$ , we have  $n = v_C(0) + v_C(1) + v_C(\times)$ . Thus,

$$\begin{aligned} v_C(0) &= n - v_C(1) - v_C(\times) \\ &= n - (e_C(1) - k) - 2k \\ &= n - e_C(1) - k. \end{aligned}$$

□

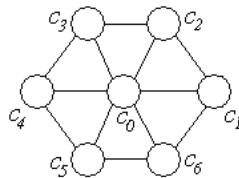
Note that since every unlabeled vertex requires one 0-edge and one 1-edge,  $v_C(\times) = 2k > 0$  unlabeled vertices require  $k$  1-edges. This leads us to

**Lemma 2.11** *In a cycle  $C_n$  with an edge labeling  $f$  (not necessarily edge-friendly) where the number of unlabeled vertices  $v_C(\times) = 2k > 0$ . We have*

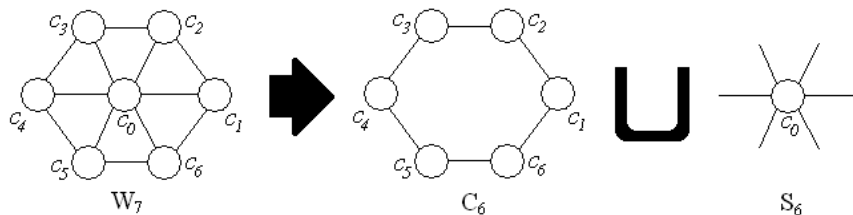
$$1 \leq k \leq e_C(1).$$

### 3 On Edge-balance Index Sets of Wheels

For an integer  $n \geq 4$ , the **wheel** on  $n$  vertices,  $W_n$ , is a graph with  $n$  vertices  $\{c_0, c_1, c_2, \dots, c_{n-1}\}$ , where  $c_0$  is of degree  $n - 1$  and all the other vertices are of degree 3. It is the cycle  $C_{n-1}$  with an additional vertex  $c_0$  connected to each vertex of  $C_{n-1}$ . The vertex  $c_0$  is called the hub, and the edges connecting the hub to the other vertices are called the spokes.  $W_7$  is displayed below:



By the nature of a wheel  $W_n$ , we split a wheel into two pieces, a cycle  $C_{n-1}$  outside and a star inside. The star part, denoted by  $S_{n-1}$ , contains the center vertex  $c_0$  and  $n - 1$  edges. Note that  $S_{n-1}$  is not a subgraph of  $W_n$ . One can see  $W_7$  here as an example.



We now use the notations from section 2. Let  $f$  be an edge-friendly labeling of  $W_n$ . We denote the number of edges of  $C_{n-1}$  which are labeled 0 and 1 by  $f$  by  $e_C(0)$  and  $e_C(1)$  and the number of edges in  $S_{n-1}$  which are labeled 0 and 1 by  $f$  by  $e_S(0)$  and  $e_S(1)$ .

$S_{n-1}$  has only one vertex. Let  $\delta$  be the value of balance index in the form of  $v(0) - v(1)$  of this vertex. Thus,

$$\delta = \begin{cases} 1 & \text{if } c_0 \text{ is labeled 0,} \\ 0 & \text{if } c_0 \text{ is not labeled,} \\ -1 & \text{if } c_0 \text{ is labeled 1.} \end{cases}$$

If we focus on  $C_{n-1}$ , then the restriction of  $f$  on  $C_{n-1}$  is an edge labeling of  $C_{n-1}$ . Thus, all the results in section 2 are applicable here. Therefore, we still denote the number of vertices  $C_{n-1}$  labeled 0, 1, and not labeled by the restricted  $f^+$  by  $v_C(0)$ ,  $v_C(1)$ , and  $v_C(x)$ , respectively.

When we put  $S_{n-1}$  into  $C_{n-1}$  to get our wheel back, the labels of vertices of  $C_{n-1}$  change. To distinguish before and after, we name the number of vertices  $C_{n-1}$  in  $W_n$  labeled 0 or 1 by the original  $f^+$  by  $v_W(0)$  or  $v_W(1)$ , respectively.

By the above notations, we have

**Lemma 3.1** *Let  $W_n$  be a wheel. The edge-balance index is*

$$v(0) - v(1) = v_W(0) - v_W(1) + \delta.$$

By the symmetry of the role of 0 and 1 in the labeling, to calculate the edge-balance index, without loss of generality, we may assume that

$$e_C(0) \geq e_C(1) \geq 0.$$

Since  $e_C(0) \geq e_C(1)$  and  $e_C(0) + e_C(1) = n - 1$ , we can find the range of possible values of  $e_C(1)$  as follow:

1.  $0 \leq e_C(1) \leq \frac{n-1}{2}$  if  $n$  is odd, or,
2.  $0 \leq e_C(1) \leq \frac{n}{2} - 1$  if  $n$  is even.

In  $C_{n-1}$ , under the assumption  $e_C(0) \geq e_C(1) \geq 0$ , there are three possible cases of the relationship between the values of  $e_C(0)$  and  $e_C(1)$ :

1.  $e_C(0) > e_C(1) \geq 1$ , or,
2.  $e_C(0) = e_C(1) \geq 1$  only when  $n$  is odd (which implies that  $e_C(0) = e_C(1) = \frac{n-1}{2}$ ), or,
3.  $e_C(1) = 0$  (which implies that  $e_C(0) = n - 1$ .)

For case 1, by Lemma 2.6,  $v_C(\times)$  is not zero. Since  $v_C(\times)$  must be even, we assume that  $v_C(\times) = 2k > 0$  where  $k$  is a positive integer. By Lemma 2.9 and Corollary 2.10, we have

$$v_C(1) = e_C(1) - k$$

and

$$v_C(0) = (n - 1) - e_C(1) - k.$$

After putting all  $S_{n-1}$  into  $C_{n-1}$  to get our wheel back, all vertices of  $C_{n-1}$  become of order 3. Thus, the Rule A and Rule B in section 2 apply. Therefore, by Rule B, all unlabeled vertices in  $C_{n-1}$  become labeled by either 0 or 1. Also, by Rule A, all other labeled vertices remain labeled by the same value.

Assume that there are  $h$  0-edges in  $S_{n-1}$  connected to an unlabeled vertex of  $C_{n-1}$ . Obviously,  $h \leq \min\{e_S(0), v_C(\times)\}$  since there are  $e_S(0)$  0-edges to be used to connect to an unlabeled vertex of  $C_{n-1}$ , and there are only  $v_C(\times)$  unlabeled vertices to choose from.

If the number of labeled vertices of  $C_{n-1}$  is less than  $e_S(0)$ , then there are some 0-edges of  $S_{n-1}$  which must connect to an unlabeled vertex of  $C_{n-1}$ . Otherwise, 0-edges of  $S_{n-1}$  may or may not connect to an unlabeled vertex of  $C_{n-1}$ . Since the number of labeled vertices of  $C_{n-1}$  is  $(n - 1) - v_C(\times)$ , there must be at least  $e_S(0) - (n - 1) + v_C(\times)$  0-edges connecting to an unlabeled vertex of  $C_{n-1}$ . Thus, we have  $\max\{0, e_S(0) - (n - 1) + v_C(\times)\} \leq h$ .

This discussion leads us to state the following:

**Lemma 3.2** *In a wheel  $W_n$ , assume that there are  $h$  0-edges in  $S_{n-1}$  connected to an unlabeled vertex of  $C_{n-1}$ . Then, we have*

$$\max\{0, e_S(0) - (n - 1) + v_C(\times)\} \leq h \leq \min\{e_S(0), v_C(\times)\}.$$

Since we have  $h$  0-vertices converted from unlabeled vertices, and the rest  $v_C(\times) - h$  unlabeled vertices are converted into vertices labeled 1, the number of vertices of  $C_{n-1}$  labeled 0 is

$$v_W(0) = v_C(0) + h$$

and the number of vertices of  $C_{n-1}$  labeled 1 is

$$v_W(1) = v_C(1) + v_C(\times) - h.$$

So, by Lemma 3.1, the edge-balance index

$$\begin{aligned} v(0) - v(1) &= (v_C(0) + h) - (v_C(1) + v_C(\times) - h) + \delta \\ &= v_C(0) + 2h - v_C(1) - 2k + \delta \end{aligned}$$

$$\begin{aligned} &= (n - 1 - e_C(1) - k) + 2h - (e_C(1) - k) - 2k + \delta \\ &= n - 1 - 2(e_C(1) + k - h) + \delta. \end{aligned}$$

With Lemmas 2.11 and 3.2, we conclude that

**Lemma 3.3** *In a wheel  $W_n$ , assume that  $v_C(\times) = 2k > 0$  where  $k$  is a positive integer and  $e_C(0) > e_C(1) \geq 1$ . Let  $h$  be the number of 0-edges in the part of  $S_{n-1}$  connected to an unlabeled vertex of  $C_{n-1}$ . Then, the edge-balance index is*

$$v(0) - v(1) = n - 1 - 2(e_C(1) + k - h) + \delta,$$

where

$$1 \leq k \leq e_C(1),$$

$$\max\{0, e_S(0) - (n - 1) + v_C(\times)\} \leq h \leq \min\{e_S(0), v_C(\times)\},$$

and

$$0 \leq e_C(1) \leq \begin{cases} \frac{n-1}{2} - 1 & \text{if } n \text{ is odd, or,} \\ \frac{n}{2} - 1 & \text{if } n \text{ is even.} \end{cases}$$

For case 2, the argument is very similar to that of case 1. By Lemma 2.9 and Corollary 2.10, we get similar equations

$$v_W(0) = v_C(0) + h$$

and

$$v_W(1) = v_C(1) + v_C(\times) - h.$$

Since  $e_C(0) = e_C(1) = \frac{n-1}{2}$ , by Lemma 3.1, the edge-balance index is

$$\begin{aligned} v(0) - v(1) &= (v_C(0) + h) - (v_C(1) + v_C(\times) - h) + \delta \\ &= v_C(0) + 2h - v_C(1) - 2k + \delta \\ &= (n - 1 - e_C(1) - k) + 2h - (e_C(1) - k) - 2k + \delta \\ &= n - 1 - 2\left(\frac{n-1}{2} + k - h\right) + \delta \\ &= 2(h - k) + \delta \end{aligned}$$

Because  $e_C(1) = \frac{n-1}{2}$ , by Lemma 2.11, we have  $1 \leq k \leq \frac{n-1}{2}$ .

For  $h$ , if  $1 \leq k \leq \frac{n-1}{4}$ , then  $e_S(0) = \frac{n-1}{2} \geq 2k = v_C(\times)$ , and  $e_S(0) - (n - 1) + v_C(\times) = \frac{n-1}{2} - (n - 1) + 2k \leq 0$ . Therefore, by Lemma 3.2, we have

$$0 \leq h \leq 2k.$$

If  $\frac{n-1}{4} \leq k \leq \frac{n-1}{2}$ , then  $e_S(0) = \frac{n-1}{2} \leq 2k = v_C(\times)$ , and  $e_S(0) - (n - 1) + v_C(\times) = \frac{n-1}{2} - (n - 1) + 2k \geq 0$ . Therefore, by Lemma 3.2, we have

$$2k - \frac{n-1}{2} \leq h \leq \frac{n-1}{2}.$$

This discussion leads us to state the following:

**Lemma 3.4** *In a wheel  $W_n$ , assume that  $v_C(\times) = 2k > 0$  where  $k$  is a positive integer and  $e_C(0) = e_C(1) \geq 1$ . Let  $h$  be the number of 0-edges in the part of  $S_{n-1}$  connected to an unlabeled vertex of  $C_{n-1}$ . Then, the edge-balance index is*

$$v(0) - v(1) = 2(h - k) + \delta,$$

where

$$1 \leq k \leq \frac{n-1}{2},$$

and

$$\begin{cases} 0 \leq h \leq 2k & \text{if } 1 \leq k \leq \frac{n-1}{4}, \text{ or,} \\ 2k - \frac{n-1}{2} \leq h \leq \frac{n-1}{2} & \text{if } \frac{n-1}{4} \leq k \leq \frac{n-1}{2}. \end{cases}$$

For case 3, since all the edges in  $C_{n-1}$  are labeled 0, we have  $n - 1$  vertices labeled 0. By Rule A, the edge-balance index is

$$v_W(0) - v_W(1) = n - 1.$$

**Lemma 3.5** *In a wheel  $W_n$  with  $v_C(\times) = 0$ , the edge-balance index is*

$$v(0) - v(1) = n - 1 + \delta,$$

Now, we are in a position to determine the edge-balance index sets of  $W_n$ .

**Theorem 3.6** *If  $n$  is even, then*

$$EBI(W_n) = \{0, 2, \dots, 2i, \dots, n - 2\}.$$

**Proof.** When  $n$  is even, we have only two cases, 1 and 3.

Since the number of edges of the wheel  $W_n$  is even, (actually, it is  $2n - 2$ ), and  $f$  is an edge-friendly labeling, we have  $e_S(0) = e_C(1)$  and  $e_S(1) = e_C(0)$ .

For case 1, by the assumption  $e_C(0) > e_C(1) \geq 1$ , we have  $e_S(1) > e_S(0)$ . This implies that  $c_0$  must be labeled 1. Thus,  $\delta = -1$ .

By Lemma 3.3 and  $\delta = -1$ , we have

$$\begin{aligned} v(0) - v(1) &= n - 1 - 2(e_C(1) + k - h) + \delta \\ &= n - 2 - 2(e_C(1) + k - h), \end{aligned}$$

where

$$1 \leq k \leq e_C(1),$$

$$\max\{0, e_S(0) - (n - 1) + v_C(\times)\} \leq h \leq \min\{e_S(0), v_C(\times)\},$$

and

$$1 \leq e_C(1) \leq \frac{n}{2} - 1.$$

Since  $n$  is even,  $v(0) - v(1) = n - 2 - 2(e_C(1) + k - h)$  must be even.

If  $v_C(\times) \geq e_S(0)$ , then we have

$$\max\{0, e_S(0) - (n - 1) + v_C(\times)\} \leq h \leq e_S(0) = e_C(1).$$

This implies that  $1 \leq e_C(1) + k - h \leq 2e_C(1) - \max\{0, e_S(0) - (n - 1) + v_C(\times)\} \leq n - 2 - \max\{0, e_C(1) - (n - 1) + 2k\}$ .

If  $v_C(\times) \leq e_S(0)$ , then  $e_S(0) - (n - 1) + v_C(\times) \leq 2e_S(0) - (n - 1) \leq 0$  since  $e_S(0) = e_C(1) \leq \frac{n}{2} - 1$ . Thus, we have

$$0 \leq h \leq v_C(\times) = 2k.$$

This implies that  $k \leq e_C(1) + k - h \leq 2e_C(1) \leq n - 2$ .

From the above inequalities, it is easy to see that we have all necessary combinations of  $k$  and  $h$  to get  $-(n - 2) \leq v(0) - v(1) \leq n - 4$ . Therefore,  $\{0, 2, \dots, 2i, \dots, n - 2\}$  is a subset of  $\text{EBI}(W_n)$ .

For case 3, by the assumption  $e_C(1) = 0$ , we have  $e_S(0) = 0$  and  $e_S(1) = n - 1$ . This implies that  $c_0$  must be labeled 1. Thus,  $\delta = -1$ . By Lemma 3.5,

$$v(0) - v(1) = n - 1 + \delta = n - 2.$$

Since when  $n$  is even, case 1 and case 3 are the only two cases, the edge-balance index set of  $W_n$  is

$$\begin{aligned} \text{EBI}(W_n) &= \{0, 2, \dots, 2i, \dots, n - 2\} \cup \{n - 2\} \\ &= \{0, 2, \dots, 2i, \dots, n - 2\}. \end{aligned}$$

□

**Example 3.7** Figure 5 shows that  $\text{EBI}(W_6) = \{0, 2, 4\}$ .

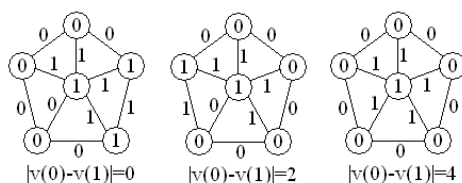


Figure 5:  $\text{EBI}(W_6)$

**Example 3.8** Figure 6 shows that  $\text{EBI}(W_8) = \{0, 2, 4, 6\}$ .

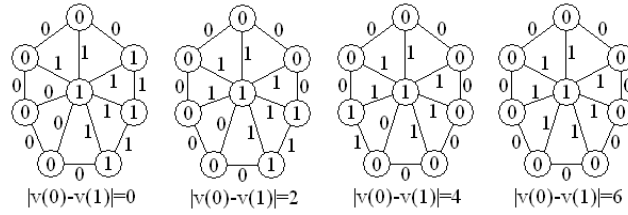


Figure 6:  $\text{EBI}(W_8)$

**Theorem 3.9** *If  $n$  is odd, then the edge-balance index set of  $W_n$  is*

1.  $\{1, 3, \dots, 2i + 1, \dots, n - 2\} \cup \{0, 2, 4, \dots, \frac{n-1}{2}\}$ , if  $n \equiv 1 \pmod{4}$ , or,
2.  $\{1, 3, \dots, 2i + 1, \dots, n - 2\} \cup \{0, 2, 4, \dots, \frac{n-1}{2} - 1\}$ , if  $n \equiv 3 \pmod{4}$ .

**Proof.** Since the number of edges of the wheel  $W_n$  is even, (actually, it is  $2n - 2$ .) and  $f$  is an edge-friendly labeling, we have  $e_S(0) = e_C(1)$  and  $e_S(1) = e_C(0)$ .

For cases 1 and 3, an argument similar to the one in Theorem 3.6 applies here. Thus, we have  $v(0) - v(1) = n - 2 - 2t$  where  $1 \leq t \leq n - 3$ , being  $t = e_C(1) + k - h$  is an integer. This implies that  $-(n - 4) \leq v(0) - v(1) \leq n - 4$ . Note that  $v(0) - v(1)$  must be odd since  $n$  is odd. Therefore,  $\text{EBI}(W_n)$  contains  $\{1, 3, \dots, 2i + 1, \dots, n - 4, n - 2\}$ . Note that  $n - 2$  comes merely from case 3.

For case 2, since  $e_C(0) = e_C(1) = \frac{n-1}{2}$ ,  $c_0$  is not labeled because  $e_S(0) = e_C(1) = \frac{n-1}{2} = e_C(0) = e_S(1)$ . By Lemma 3.4, we have  $v(0) - v(1) = 2(h - k)$  where  $1 \leq k \leq \frac{n-1}{2}$  and  $0 \leq h \leq 2k$  if  $1 \leq k \leq \frac{n-1}{4}$  and  $2k - \frac{n-1}{2} \leq h \leq \frac{n-1}{2}$  if  $\frac{n-1}{4} \leq k \leq \frac{n-1}{2}$ . Thus, in this case, the edge-balance index must be even. Even with the above inequalities, it is easy to see that we have all necessary combinations of  $k$  and  $h$  to get  $-\frac{n-1}{2} \leq v(0) - v(1) = 2(h - k) \leq \frac{n-1}{2}$ . Therefore,  $\text{EBI}(W_n)$  contains  $\{0, 2, 4, \dots, \frac{n-1}{2}\}$  if  $n \equiv 1 \pmod{4}$ , or,  $\{0, 2, 4, \dots, \frac{n-1}{2} - 1\}$  if  $n \equiv 3 \pmod{4}$ .

With all three cases together, the edge-balance index set of  $W_n$  is

1.  $\{1, 3, \dots, 2i + 1, \dots, n - 2\} \cup \{0, 2, 4, \dots, \frac{n-1}{2}\}$ , if  $n \equiv 1 \pmod{4}$ , or,
2.  $\{1, 3, \dots, 2i + 1, \dots, n - 2\} \cup \{0, 2, 4, \dots, \frac{n-1}{2} - 1\}$ , if  $n \equiv 3 \pmod{4}$ .

□

**Example 3.10** We list the edge-balance index set of  $W_n$  when  $n$  is odd and small.

1.  $\text{EBI}(W_5) = \{0, 1, 2, 3\}$ ;
2.  $\text{EBI}(W_7) = \{0, 1, 2, 3, 5\}$ ;

3.  $\text{EBI}(W_9) = \{0, 1, 2, 3, 4, 5, 7\}$ ;
4.  $\text{EBI}(W_{11}) = \{0, 1, 2, 3, 4, 5, 7, 9\}$ ;
5.  $\text{EBI}(W_{13}) = \{0, 1, 2, 3, 4, 5, 6, 7, 9, 11\}$ ;
6.  $\text{EBI}(W_{15}) = \{0, 1, 2, 3, 4, 5, 6, 7, 9, 11, 13\}$ ;
7.  $\text{EBI}(W_{17}) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 15\}$ .

Finally we present, for illustrative purposes, labeled graphs for  $n = 7$  and 11 which give rise to their edge-balance index sets.

**Example 3.11** Figure 7 shows that  $\text{EBI}(W_7) = \{0, 1, 2, 3, 5\}$ . Note that 4 is missing.

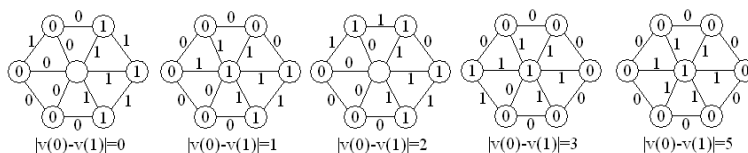


Figure 7:  $\text{EBI}(W_7)$

**Example 3.12** Figure 8 shows that  $\text{EBI}(W_{11}) = \{0, 1, 2, 3, 4, 5, 7, 9\}$ . Note that 6 and 8 are missing.

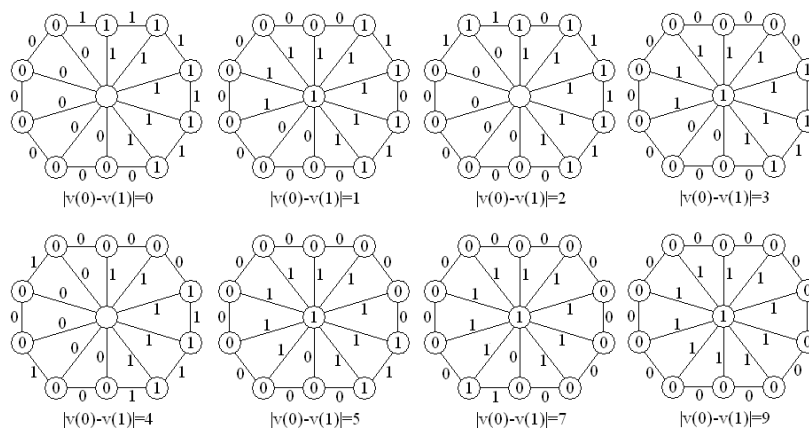


Figure 8:  $\text{EBI}(W_{11})$

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