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M/G/1 Queue with Two-Stage Heterogeneous Service Compulsory Server Vacation and Random Breakdowns

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Abstract

We analyze a single server queue with Poisson arrivals, two stages of heterogeneous service with different (arbitrary) service time distributions subject to random breakdowns and compulsory server vacations with general (arbitrary) vacation periods. After first-stage service the server must provide the second stage service. However, after the completion of each second stage service, the server will take compulsory vacation. The system may breakdown at random and repair time follow exponential distribution. The time dependent probability generating functions have been obtained in terms of their Laplace transforms and the corresponding steady state results have been obtained explicitly. Also the average number of customers in the queue and the average waiting time are derived.

Mathematics Subject Classification: 60K25, 60K30

Keywords: Poisson arrivals, probability generating function, idle state, steady state, mean queue length, mean waiting time

1 Introduction

Vacation queues have been studied extensively by numerous authors including Levy and Yechiali [10], Doshi [5] and Madan [11], [12] and [13] due to their various applications in Communication systems, Computer network and etc.

Chae et al. [3], Chang and Takine [4] and Igaki [7] have studied queues with generalized vacations. Vacation queues with c servers have been studied by Tian et al. [15]. Choudhury and Borthakur [2] have studied vacation queues with batch arrivals. Multiple vacations have been studied by Tian and Zhang [17].

In this paper, we consider queueing system subject to compulsory server vacation and random breakdowns. A queueing system might suddenly break down and hence the server will not be able to continue providing service unless the system is repaired. Aissani and Artalejo [1], Takine and Sengupta [16], Federgruen and So [6], Vinck and Bruneel [18] have studied different queueing systems subject to random breakdowns. Jayawardene and Kella [8] have studied $M/G/\infty$ queues with altering renewal breakdowns. Kulkarni and Choi [9] and Wang et al [19] have studied retrial queues with system breakdowns and repairs.

Madan and Maraghi [14] have studied batch arrival queueing system with random breakdowns and Bernoulli schedule server vacations having general vacation time. They have obtained steady state results in terms of the probability generating functions for the number of customers in the queue.

In this paper we consider queueing system with compulsory server vacation and random breakdowns. Each arriving customer has to under go two stages of service provided by a single server and the service time for two stages are assumed to follow general distribution. As soon as the second stage of a customer's service is complete, the server will go for compulsory vacation. The vacation times are also assumed to be general while we consider exponential distribution for repair time. And once the system break down, it enters a repair process and the customer whose service is interrupted goes back to the head of the queue where the arrivals are Poisson. The customers arrive to the system one by one and are served on a first come-first served basis.

The rest of the paper is organized as follows. The mathematical description of our model is in Section 2 and equations governing the model are given in Section 3. The time dependent solution have been obtained in Section 4 and the corresponding steady state results have been derived explicitly in Section 5. Mean queue length and mean waiting time are computed in Section 6 and in Section 7 respectively.

2 Mathematical Description of the model

We assume the following to describe the queueing model of our study.

- Customers arrive at the system one by one in according to a Poisson stream with arrival rate $\lambda(> 0)$.

- Each customer undergoes two stages of heterogeneous service provided by a single server on a first come first served basis. The service time of the two stages follow different general (arbitrary) distributions with distribution function $B_j(v)$ and the density function $b_j(v),\ j = 1, 2$

- Let $\mu_i(x)\mathrm{d}x$ be the conditional probability of completion of the i^{th} stage of service during the interval $(x, x + \mathrm{d}x]$ given that elapsed time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)},\ i = 1, 2, \tag{1}$$

  and therefore,

$$b_i(v) = \mu_i(v)e^{-\int\limits_0^v \mu_i(x)\mathrm{d}x},\ i = 1, 2. \tag{2}$$

- As soon as the second stage of a customers's service is complete, the server will take compulsory vacation of random length.

- The vacation time also follow general (arbitrary) distribution with distribution function $V(s)$ and the density function $v(s)$. Let $\gamma(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x + \mathrm{d}x]$ given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{v(x)}{1 - V(x)} \tag{3}$$

  and therefore,

$$v(s) = \gamma(s)e^{-\int\limits_0^s \gamma(x)\mathrm{d}x}. \tag{4}$$

- On returning from vacation the server instantly starts serving the customer at the head of the queue, if any.

- The customers are served according to the first come, first served rule.

- The system may break down at random and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$. Further we assume that once the system breaks down, the customer whose service is interrupted comes back to the head of the queue.

- Once the system breaks down, it enters a repair process immediately. The repair times are exponentially distributed with mean repair rate $\beta > 0$.
- Various stochastic processes involved in the system are assumed to be independent of each other.

3 Definitions and equations governing the system

We define

$P_n^{(1)}(x,t)$= Probability that at time t, the server is active providing first stage of service and there are $n(\geq 0)$ customers in the queue excluding the one being served and the elapsed service time for this customer is x. Consequently $P_n^{(1)}(t) = \int\limits_0^\infty P_n^{(1)}(x,t)\mathrm{d}x$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the first stage of service irrespective of the value of x.

$P_n^{(2)}(x,t)$= Probability that at time t, the server is active providing second stage of service and there are $n(\geq 0)$ customers in the queue excluding the one being served and the elapsed service time for this customer is x. Consequently $P_n^{(2)}(t) = \int\limits_0^\infty P_n^{(2)}(x,t)\mathrm{d}x$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the second stage of service irrespective of the value of x.

$V_n(x,t)$=Probability that at time t, the server is under vacation with elapsed vacation time x and there are $n(\geq 0)$ customers waiting in the queue for service. Consequently $V_n(t) = \int\limits_0^\infty V_n(x,t)\mathrm{d}x$ denotes the probability that at time t there are n customers in the queue and the server is under vacation irrespective of the value of x.

$R_n(t)$=Probability that at time t, the server is inactive due to system breakdown and the system is under repair, while there are $n(n \geq 0)$ customers in the queue.

$Q(t)$=Probability that at time t, there are no customers in the system and the server is idle but available in the system.

The model is then, governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t) + \frac{\partial}{\partial t}P_n^{(1)}(x,t) + (\lambda + \mu_1(x) + \alpha)P_n^{(1)}(x,t) = \lambda P_{n-1}^{(1)}(x,t), \quad n = 1,2,\ldots \tag{5}$$

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t)+\frac{\partial}{\partial t}P_0^{(1)}(x,t)+(\lambda+\mu_1(x)+\alpha)P_0^{(1)}(x,t) = 0, \tag{6}$$

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t)+\frac{\partial}{\partial t}P_n^{(2)}(x,t)+(\lambda+\mu_2(x)+\alpha)P_n^{(2)}(x,t) = \lambda P_{n-1}^{(2)}(x,t), \quad n=1,2,\ldots \tag{7}$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t)+\frac{\partial}{\partial t}P_0^{(2)}(x,t)+(\lambda+\mu_2(x)+\alpha)P_0^{(2)}(x,t) = 0, \tag{8}$$

$$\frac{\partial}{\partial x}V_n(x,t)+\frac{\partial}{\partial t}V_n(x,t)+(\lambda+\gamma(x))V_n(x,t) = \lambda V_{n-1}(x,t), \quad n=1,2,\ldots \tag{9}$$

$$\frac{\partial}{\partial x}V_0(x,t)+\frac{\partial}{\partial t}V_0(x,t)+(\lambda+\gamma(x))V_0(x,t) = 0, \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}R_n(t) = -(\lambda+\beta)R_n(t)+\lambda R_{n-1}(t)+\alpha\int_0^\infty P_{n-1}^{(1)}(x,t)\mathrm{d}x+\alpha\int_0^\infty P_{n-1}^{(2)}(x,t)\mathrm{d}x, \quad n=1,2,\ldots \tag{11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}R_0(t) = -(\lambda+\beta)R_0(t), \tag{12}$$

$$\frac{d}{dt}Q(t) = -\lambda Q(t)+R_0(t)\beta+\int_0^\infty V_0(x,t)\gamma(x)\mathrm{d}x\ , \tag{13}$$

Equations (5)-(13) are to be solved subject to the following boundary conditions:

$$P_0^{(1)}(0,t) = Q(t)\lambda+R_1(t)\beta+\int_0^\infty V_1(x,t)\gamma(x)\mathrm{d}x, \tag{14}$$

$$P_n^{(1)}(0,t) = R_{n+1}(t)\beta+\int_0^\infty V_{n+1}(x,t)\gamma(x)\mathrm{d}x,\ n=1,2,\ldots, \tag{15}$$

$$P_n^{(2)}(0,t) = \int_0^\infty P_n^{(1)}(x,t)\mu_1(x)\mathrm{d}x,\ n=0,1,\ldots, \tag{16}$$

$$V_n(0,t) = \int_0^\infty P_n^{(2)}(x,t)\mu_2(x)\mathrm{d}x,\ n=0,1,\ldots \tag{17}$$

We assume that initially there is no customer in the system and the server is idle. So the initial conditions are

$$\begin{aligned} V_0(0) = V_n(0) = 0,\ Q(0) = 1 \ \text{ and} \\ P_n^j(0) = 0 \ \text{ for } \ n = 0, 1, 2, \dots, j = 1, 2. \end{aligned} \tag{18}$$

4 Generating functions of the queue length: The time-dependent solution

In this section we obtain the transient solution for the above set of differential-difference equations.

Theorem 4.1 The system of differential difference equations to describe an $M/G/1$ queue with two stages of heterogeneous service subject to compulsory server vacation and random breakdowns are given by equations (5)-(17) with initial conditions (18) and the generating functions of transient solution are given by equations (61)-(64).

Proof We define the probability generating functions,

$$\left.\begin{aligned} P_q^{(1)}(x,z,t) &= \sum_{n=0}^{\infty} z^n P_n^{(1)}(x,t), \\ P_q^{(1)}(z,t) &= \sum_{n=0}^{\infty} z^n P_n^{(1)}(t), \\ P_q^{(2)}(x,z,t) &= \sum_{n=0}^{\infty} z^n P_n^{(2)}(x,t), \\ P_q^{(2)}(z,t) &= \sum_{n=0}^{\infty} z^n P_n^{(2)}(t), \\ V_q(x,z,t) &= \sum_{n=0}^{\infty} z^n V_n(x,t), \\ V_q(z,t) &= \sum_{n=0}^{\infty} z^n V_n(t), \\ R_q(z,t) &= \sum_{n=0}^{\infty} z^n R_n(t). \end{aligned}\right\} \tag{19}$$

which are convergent inside the circle given by $|z| \leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\overline{f}(s) = \int_0^{\infty} e^{-st} f(t)\mathrm{d}t,\ \Re(s) > 0. \tag{20}$$

Taking the Laplace transforms of equations (5) to (17) and using (18), we obtain

$$\frac{\partial}{\partial x}\overline{P}_n^{(1)}(x,s)+(s+\lambda+\mu_1(x)+\alpha)\overline{P}_n^{(1)}(x,s) = \lambda\overline{P}_{n-1}^{(1)}(x,s), n=1,2,\ldots \tag{21}$$

$$\frac{\partial}{\partial x}\overline{P}_0^{(1)}(x,s)+(s+\lambda+\mu_1(x)+\alpha)\overline{P}_0^{(1)}(x,s) = 0, \tag{22}$$

$$\frac{\partial}{\partial x}\overline{P}_n^{(2)}(x,s)+(s+\lambda+\mu_2(x)+\alpha)\overline{P}_n^{(2)}(x,s) = \lambda\overline{P}_{n-1}^{(2)}(x,s), n=1,2,\ldots \tag{23}$$

$$\frac{\partial}{\partial x}\overline{P}_0^{(2)}(x,s)+(s+\lambda+\mu_2(x)+\alpha)\overline{P}_0^{(2)}(x,s) = 0, \tag{24}$$

$$\frac{\partial}{\partial x}\overline{V}_n(x,s)+(s+\lambda+\gamma(x))\overline{V}_n(x,s) = \lambda\overline{V}_{n-1}(x,s), n=1,2,\ldots \tag{25}$$

$$\frac{\partial}{\partial x}\overline{V}_0(x,s)+(s+\lambda+\gamma(x))\overline{V}_0(x,s) = 0, \tag{26}$$

$$(s+\lambda+\beta)\overline{R}_n(s) = \lambda\overline{R}_{n-1}(s)+\alpha\int_0^\infty\overline{P}_{n-1}^{(1)}(x,s)\mathrm{d}x+\alpha\int_0^\infty\overline{P}_{n-1}^{(2)}(x,s)\mathrm{d}x, \quad n=1,2,\ldots \tag{27}$$

$$(s+\lambda+\beta)\overline{R}_0(s) = 0, \tag{28}$$

$$(s+\lambda)\overline{Q}(s) = 1+\overline{R}_0(s)+\int_0^\infty\overline{V}_0(x,s)\gamma(x)\mathrm{d}x, \tag{29}$$

$$\overline{P}_0^{(1)}(0,s) = \overline{Q}(s)\lambda+\overline{R}_1(s)\beta+\int_0^\infty\overline{V}_1(x,s)\gamma(x)\mathrm{d}x, \tag{30}$$

$$\overline{P}_n^{(1)}(0,s) = \overline{R}_{n+1}(s)\beta+\int_0^\infty\overline{V}_{n+1}(x,s)\gamma(x)\mathrm{d}x,\ n=1,2,\ldots \tag{31}$$

$$\overline{P}_n^{(2)}(0,s) = \int_0^\infty\overline{P}_n^{(1)}(x,s)\mu_1(x)\mathrm{d}x,\ n=0,1,\ldots \tag{32}$$

$$\overline{V}_n(0,s) = \int_0^\infty\overline{P}_n^{(2)}(x,s)\mu_2(x)\mathrm{d}x,\ n=0,1,\ldots. \tag{33}$$

Now multiplying equation (21) by z^n and summing over n from 1 to ∞, adding to equation (22) and using the generating functions defined in (19), we get

$$\frac{\partial}{\partial x}\overline{P}_q^{(1)}(x,z,s) + (s+\lambda-\lambda z+\mu_1(x)+\alpha)\overline{P}_q^{(1)}(x,z,s) = 0. \tag{34}$$

Performing similar operations on equations (23) to (28) we obtain

$$\frac{\partial}{\partial x}\overline{P}_q^{(2)}(x,z,s) + (s+\lambda-\lambda z+\mu_2(x)+\alpha)\overline{P}_q^{(2)}(x,z,s) = 0, \tag{35}$$

$$\frac{\partial}{\partial x}\overline{V}_q(x,z,s) + (s+\lambda-\lambda z+\gamma(x))\overline{V}_q(x,z,s) = 0. \tag{36}$$

$$(s+\lambda-\lambda z+\beta)\overline{R}_q(z,s) = \alpha z\left[\int_0^\infty \overline{P}_q^{(1)}(x,z,s)\mathrm{d}x + \int_0^\infty \overline{P}_q^{(2)}(x,z,s)\mathrm{d}x\right]. \tag{37}$$

For the boundary conditions, we multiply both sides of equation (30) by z, multiply both sides of equation (31) by z^{n+1}, sum over n from 1 to ∞, add the two results and use equation (19) to get

$$z\overline{P}_q^{(1)}(0,z,s) = \lambda z\overline{Q}(s) + \beta\overline{R}_q(z,s) - \beta\overline{R}_0(s) + \int_0^\infty \overline{V}_q(x,z,s)\gamma(x)\mathrm{d}x - \int_0^\infty \overline{V}_0(x,s)\gamma(x)\mathrm{d}x. \tag{38}$$

Performing similar operations on equations (32) and (33), we obtain

$$\overline{P}_q^{(2)}(0,z,s) = \int_0^\infty \overline{P}_q^{(1)}(x,z,s)\mu_1(x)\mathrm{d}x. \tag{39}$$

$$\overline{V}_q(0,z,s) = \int_0^\infty \overline{P}_q^{(2)}(x,z,s)\mu_2(x)\mathrm{d}x. \tag{40}$$

Using equation (29), equation (38) becomes

$$z\overline{P}_q^{(1)}(0,z,s) = (1-s\overline{Q}(s)) + \lambda(z-1)\overline{Q}(s) + \beta\overline{R}_q(z,s) + \int_0^\infty \overline{V}_q(x,z,s)\gamma(x)\mathrm{d}x. \tag{41}$$

Integrating equation (34) from 0 to x yields

$$\overline{P}_q^{(1)}(x,z,s) = \overline{P}_q^{(1)}(0,z,s)\ e^{-(s+\lambda-\lambda z+\alpha)x-\int\limits_0^x \mu_1(t)dt}, \tag{42}$$

where $\overline{P}_q^{(1)}(0,z,s)$ is given by equation (41). Again integrating equation (42) by parts with respect to x yields

$$\overline{P}_q^{(1)}(z,s) = \overline{P}_q^{(1)}(0,z,s)\left[\frac{1-\overline{B}_1(s+\lambda-\lambda z+\alpha)}{s+\lambda-\lambda z+\alpha}\right]. \tag{43}$$

where

$$\overline{B}_1(s+\lambda-\lambda z+\alpha) = \int\limits_0^\infty e^{-(s+\lambda-\lambda z+\alpha)x}dB_1(x) \tag{44}$$

is the Laplace-Stieltjes transform of the first stage service time $B_1(x)$. Now multiplying both sides of equation (42) by $\mu_1(x)$ and integrating over x we obtain

$$\int\limits_0^\infty \overline{P}_q^{(1)}(x,z,s)\mu_1(x)\mathrm{d}x = \overline{P}_q^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda z+\alpha). \tag{45}$$

Similarly, on integrating equations (35) and (36) from 0 to x, we get

$$\overline{P}_q^{(2)}(x,z,s) \;=\; \overline{P}_q^{(2)}(0,z,s)\ e^{-(s+\lambda-\lambda z+\alpha)x-\int\limits_0^x \mu_2(t)dt}, \tag{46}$$

$$\overline{V}_q(x,z,s) \;=\; \overline{V}_q(0,z,s)\ e^{-(s+\lambda-\lambda z)x-\int\limits_0^x \gamma(t)dt}. \tag{47}$$

where $\overline{P}_q^{(2)}(0,z,s)$ and $\overline{V}_q(0,z,s)$ are given by equations (39) and (40). Again integrating equations (46) and (47) by parts with respect to x yields

$$\overline{P}_q^{(2)}(z,s) \;=\; \overline{P}_q^{(2)}(0,z,s)\left[\frac{1-\overline{B}_2(s+\lambda-\lambda z+\alpha)}{s+\lambda-\lambda z+\alpha}\right], \tag{48}$$

$$\overline{V}_q(z,s) \;=\; \overline{V}_q(0,z,s)\left[\frac{1-\overline{V}(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right]. \tag{49}$$

where

$$\overline{B}_2(s+\lambda-\lambda z+\alpha) = \int\limits_0^\infty e^{-(s+\lambda-\lambda z+\alpha)x}dB_2(x) \tag{50}$$

is the Laplace-Stieltjes transform of the second stage service time $B_2(x)$. Now multiplying both sides of equation (46) by $\mu_2(x)$ and integrating over x we obtain

$$\int_0^\infty \overline{P}_q^{(2)}(x,z,s)\mu_2(x)\mathrm{d}x = \overline{P}_q^{(2)}(0,z,s)\overline{B}_2(s+\lambda-\lambda z+\alpha) \tag{51}$$

and

$$\overline{V}(s+\lambda-\lambda z) = \int_0^\infty e^{-(s+\lambda-\lambda z)x}dV(x) \tag{52}$$

is the Laplace-Stieltjes transform of the vacation time $V(x)$. Now multiplying both sides of equation (47) by $\gamma(x)$ and integrating over x we obtain

$$\int_0^\infty \overline{V}_q(x,z,s)\gamma(x)\mathrm{d}x = \overline{V}_q(0,z,s)\overline{V}(s+\lambda-\lambda z). \tag{53}$$

Now using equations (39), (45) and (51), we can write equation (40) as

$$\overline{V}_q(0,z,s) = \overline{P}_q^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda z+\alpha)\overline{B}_2(s+\lambda-\lambda z+\alpha). \tag{54}$$

Using above equation, equation (49) becomes

$$\overline{V}_q(z,s) = \overline{P}_q^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda z+\alpha)\overline{B}_2(s+\lambda-\lambda z+\alpha) \left[\frac{1-\overline{V}(s+\lambda-\lambda z)}{(s+\lambda-\lambda z)}\right]. \tag{55}$$

By using equation (54), equation (53) becomes

$$\int_0^\infty \overline{V}_q(x,z,s)\gamma(x)\mathrm{d}x = \overline{P}_q^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda z+\alpha)\overline{B}_2(s+\lambda-\lambda z+\alpha) \overline{V}(s+\lambda-\lambda z). \tag{56}$$

Now using equation (45), equation (39) reduces to

$$\overline{P}_q^{(2)}(0,z,s) = \overline{P}_q^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda z+\alpha). \tag{57}$$

Using equations (45) and (57), equation (37) becomes

$$\overline{R}_q(z,s) = \alpha z\overline{P}_q^{(1)}(0,z,s)\left[\frac{1-\overline{B}_1(s+\lambda-\lambda z+\alpha)\overline{B}_2(s+\lambda-\lambda z+\alpha)}{(s+\lambda-\lambda z+\alpha)(s+\lambda-\lambda z+\beta)}\right]. \tag{58}$$

Now using equations (53) and (58) in equation (41) and solving for $\overline{P}_q(0,z,s)$ we get

$$\overline{P}_q^{(1)}(0,z,s) = \frac{f_1(z)f_2(z)\left[(1-s\overline{Q}(s))+\lambda(z-1)\overline{Q}(s)\right]}{DR} \tag{59}$$

where

$$\begin{aligned} DR = f_1(z)f_2(z)&\left\{z-\overline{B}_1[f_1(z)]\overline{B}_2[f_1(z)]\overline{V}(s+\lambda-\lambda z)\right\} \\ &-\beta\alpha z\left\{1-\overline{B}_1[f_1(z)]\overline{B}_2[f_1(z)]\right\}, \end{aligned} \tag{60}$$

$$f_1(z) = s+\lambda-\lambda z+\alpha \text{ and } f_2(z) = s+\lambda-\lambda z+\beta.$$

Substituting the value of $\overline{P}_q^{(1)}(0,z,s)$ from equation (59) into equations (43), (48), (55) and (58) we get

$$\overline{P}_q^{(1)}(z,s) = \frac{f_2(z)\left[(1-s\overline{Q}(s))+\lambda(z-1)\overline{Q}(s)\right]\left[1-\overline{B}_1[f_1(z)]\right]}{DR}, \tag{61}$$

$$\overline{P}_q^{(2)}(z,s) = \frac{f_2(z)\left[(1-s\overline{Q}(s))+\lambda(z-1)\overline{Q}(s)\right]\overline{B}_1[f_1(z)]\left[1-\overline{B}_2[f_1(z)]\right]}{DR}, \tag{62}$$

$$\overline{V}_q(z,s) = \frac{f_1(z)f_2(z)\left[(1-s\overline{Q}(s))+\lambda(z-1)\overline{Q}(s)\right]\overline{B}_1[f_1(z)]\overline{B}_2[f_1(z)]}{DR} \left[\frac{1-\overline{V}(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right], \tag{63}$$

$$\overline{R}_q(z,s) = \frac{\alpha z\left[(1-s\overline{Q}(s))+\lambda(z-1)\overline{Q}(s)\right]\left[1-\overline{B}_1[f_1(z)]\overline{B}_2[f_1(z)]\right]}{DR}. \tag{64}$$

where DR is given by equation (60). Thus $\overline{P}_q^{(1)}(z,s)$, $\overline{P}_q^{(2)}(z,s)$, $\overline{V}_q(z,s)$ and $\overline{R}_q(z,s)$ are completely determined from equations (61)-(64) which completes the proof of the theorem.

5 The steady state results

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, we supress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the well-known Tauberian property,

$$\lim_{s\to 0} s\overline{f}(s) = \lim_{t\to\infty} f(t). \tag{65}$$

In order to determine $\overline{P}_q^{(1)}(z,s)$, $\overline{P}_q^{(2)}(z,s)$, $\overline{V}_q(z,s)$ and $\overline{R}_q(z,s)$ completely, we have yet to determine the unknown Q which appears in the numerators of the right hand sides of equations (61)-(64). For that purpose, we shall use the normalizing condition

$$P_q^{(1)}(1) + P_q^{(2)}(1) + V_q(1) + R_q(1) + Q = 1. \tag{66}$$

Theorem 5.1 The steady state probabilities for an $M/G/1$ queue with two stages of heterogeneous service, following general distribution subject to compulsory server vacation and random breakdowns are given by

$$\begin{aligned}
P_q^{(1)}(1) &= \frac{\beta\lambda\left[1-\overline{B}_1(\alpha)\right]Q}{dr}, && (67)\\
P_q^{(2)}(1) &= \frac{\beta\lambda\overline{B}_1(\alpha)\left[1-\overline{B}_2(\alpha)\right]Q}{dr}, && (68)\\
V_q(1) &= \frac{\lambda E(v)\alpha\beta\overline{B}_1(\alpha)\overline{B}_2(\alpha)Q}{dr}, && (69)\\
R_q(1) &= \frac{\alpha\lambda\left[1-\overline{B}_1(\alpha)\overline{B}_2(\alpha)\right]Q}{dr}, && (70)
\end{aligned}$$

where

$$dr = \alpha\beta\overline{B}_1(\alpha)\overline{B}_2(\alpha) - (\alpha+\beta)\lambda\left[1-\overline{B}_1(\alpha)\overline{B}_2(\alpha)\right] - \alpha\beta\lambda E(v)\overline{B}_1(\alpha)\overline{B}_2(\alpha), \tag{71}$$

and

$$Q = 1 - \lambda\left[\frac{1}{\beta\overline{B}_1(\alpha)\overline{B}_2(\alpha)} + \frac{1}{\alpha\overline{B}_1(\alpha)\overline{B}_2(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha} + E(v)\right] \tag{72}$$

where $P_q^{(1)}(1)$, $P_q^{(2)}(1)$, $V_q(1)$, $R_q(1)$ and Q are the steady state probabilities that the server is providing first stage of service, second stage of service, server under vacation, server under repair and the server being idle respectively without regard to the number of customers in the system.

Proof Multiplying both sides of equations (61), (62), (63) and (64) by s, taking limit as $s \to 0$, applying property (65) and simplifying, we obtain

$$\begin{aligned}
P_q^{(1)}(z) &= \frac{f_2(z)\left(1-\overline{B}_1[f_1(z)]\right)\lambda[z-1]Q}{DR}, && (73)\\
P_q^{(2)}(z) &= \frac{f_2(z)\overline{B}_1[f_1(z)]\left(1-\overline{B}_2[f_1(z)]\right)\lambda[z-1]Q}{DR}, && (74)\\
V_q(z) &= \frac{f_1(z)f_2(z)\overline{B}_1[f_1(z)]\overline{B}_2[f_1(z)]\left[\overline{V}(\lambda-\lambda z)-1\right]Q}{DR}, && (75)\\
R_q(z) &= \frac{\lambda\left[\alpha z(z-1)\right]\left\{1-\overline{B}_1[f_1(z)]\overline{B}_2[f_1(z)]\right\}Q}{DR}. && (76)
\end{aligned}$$

where DR is given by equation (60), $f_1(z)$ and $f_2(z)$ are as given in previous section. Let $W_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system. Then adding equations (61), (62) (63) and (64) we obtain

$$\begin{aligned} W_q(z) &= P_q^{(1)}(z) + P_q^{(2)}(z) + V_q(z) + R_q(z), \\ &= \frac{f_2(z)\left(1-\overline{B}_1[f_1(z)]\right)\lambda[z-1]Q}{DR} \\ &\quad + \frac{f_2(z)\overline{B}_1[f_1(z)]\left(1-\overline{B}_2[f_1(z)]\right)\lambda[z-1]Q}{DR} \\ &\quad + \frac{f_1(z)f_2(z)\overline{B}_1[f_1(z)]\overline{B}_2[f_1(z)]\left[\overline{V}(\lambda-\lambda z)-1\right]Q}{DR} \\ &\quad + \frac{\lambda\left[\alpha z(z-1)\right]\left\{1-\overline{B}_1[f_1(z)]\overline{B}_2[f_1(z)]\right\}Q}{DR}. \end{aligned} \tag{77}$$

We see that for $z=1$, $W_q(z)$ is indeterminate of the form $0/0$. Therefore, we apply L'Hôpital's rule and on simplifying we obtain the results (67) to (70), where $\overline{B}_1[0]=0, \overline{B}_2[0]=0, \overline{V}[0]=0$ and $-\overline{V}'[0]=E(v)$, the mean vacation time and

$$W_q(1) = \frac{\lambda Q\left\{(\alpha+\beta)\left[1-\overline{B}_1(\alpha)\overline{B}_1(\alpha)\right]+\alpha\beta E(v)\overline{B}_1(\alpha)\overline{B}_2(\alpha)\right\}}{dr}\ , \tag{78}$$

where dr is given by equation (71). Therefore adding Q to equation (78), equating to 1 and simplifying, we get

$$Q = 1-\lambda\left[\frac{1}{\beta\overline{B}_1(\alpha)\overline{B}_2(\alpha)}+\frac{1}{\alpha\overline{B}_1(\alpha)\overline{B}_2(\alpha)}-\frac{1}{\beta}-\frac{1}{\alpha}+E(v)\right] \tag{79}$$

And hence the utilization factor ρ of the system is given by

$$\rho = \lambda\left(\frac{1}{\beta\overline{B}_1(\alpha)\overline{B}_2(\alpha)}+\frac{1}{\alpha\overline{B}_1(\alpha)\overline{B}_2(\alpha)}-\frac{1}{\beta}-\frac{1}{\alpha}+E(v)\right). \tag{80}$$

where $\rho<1$ is the stability condition under which the steady state exists. Equation (79) gives the probability that the server is idle. Substituting for Q from (79) into (77), we have completely and explicitly determined $W_q(z)$, the probability generating function of the queue size.

6 The average queue size and the average waiting time

Let L_q denote the mean number of customers in the queue under the steady state. Then

$$L_q = \frac{\mathrm{d}}{\mathrm{d}z}W_q(z)\bigg|_{z=1}$$

Since this formula gives $0/0$ form, then we write $W_q(z)$ given in (77) as $W_q(z) = \frac{N(z)}{D(z)}$ where $N(z)$ and $D(z)$ are the numerator and denominator of the right hand side of (77) respectively. Then we use

$$\begin{aligned}
L_q &= \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}P_q(z),\\
&= P_q^{'}(1),\\
&= \lim_{z\to 1}\frac{D^{'}(z)N^{''}(z) - N^{'}(z)D^{''}(z)}{2(D^{'}(z))^2},\\
&= \lim_{z\to 1}\frac{D^{'}(1)N^{''}(1) - N^{'}(1)D^{''}(1)}{2(D^{'}(1))^2}. \qquad (81)
\end{aligned}$$

where primes and double primes in equation (81) denote the first and second derivative at $z = 1$. Carrying out the derivatives at $z = 1$, we have

$$\begin{aligned}
N^{'}(1) &= \lambda Q\left[(\alpha+\beta) + \overline{B}_1(\alpha)\overline{B}_2(\alpha)(\alpha\beta E(v) - \alpha - \beta)\right], \qquad (82)\\
N^{''}(1) &= 2Q\lambda^2\left[\left(-1+\frac{\alpha}{\lambda}\right) + \overline{B}_1(\alpha)\overline{B}_2(\alpha)\left(1-\frac{\alpha}{\lambda} - \alpha E(v) - \beta E(v)\right.\right.\\
&\quad \left.+\frac{1}{2}\alpha\beta E(v^2)\right) + \overline{B}_1^{'}(\alpha)\left[\alpha+\beta-\alpha\beta E(v)\right]\\
&\quad \left.+\overline{B}_2^{'}(\alpha)\left[\alpha+\beta-\alpha\beta E(v)\right]\right], \qquad (83)\\
D^{'}(1) &= -\lambda(\alpha+\beta) + \overline{B}_1(\alpha)\overline{B}_2(\alpha)\left[\alpha\beta + \lambda(\alpha+\beta) - \alpha\beta E(v)\right], \qquad (84)\\
D^{''}(1) &= 2\lambda^2\left[\left(1-\frac{\alpha+\beta}{\lambda}\right) + \overline{B}_1(\alpha)\overline{B}_2(\alpha)\left(-1+\alpha E(v) + \beta E(v)\right.\right.\\
&\quad \left.-\frac{1}{2}\alpha\beta E(v^2)\right) + \overline{B}_1^{'}(\alpha)\left[-\frac{\alpha\beta}{\lambda} - \alpha - \beta + \alpha\beta E(v)\right]\\
&\quad \left.+\overline{B}_2^{'}(\alpha)\left[-\frac{\alpha\beta}{\lambda} - \alpha - \beta + \alpha\beta E(v)\right]\right]. \qquad (85)
\end{aligned}$$

where $E(v^2)$ is the second moment of the vacation time and Q has been found in (72). Then if we substitute the values of $N^{'}(1), N^{''}(1), D^{'}(1)$ and $D^{''}(1)$ from equations (82) to (85) into equation (81) we obtain L_q in closed form. Further, we find the average system size L using Little's formula. Thus we have

$$L = L_q + \rho \qquad (86)$$

where L_q has been found in equation (81) and ρ is obtained from equation (80) as

$$\rho = 1 - Q. \qquad (87)$$

7 The mean waiting time

Let W_q and W denote the mean waiting time in the queue and the system respectively. Then using Little's formula, we obtain,

$$W_q = \frac{L_q}{\lambda} \tag{88}$$

$$W = \frac{L}{\lambda} \tag{89}$$

where L_q and L have been found in equations (81) and (86).

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