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Multiplicative Coupled Fibonacci Sequences and Some Fundamental Properties

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Abstract

In the recent years, there has been much interest in development of knowledge in the general region of Fibonacci numbers and related mathematical topics. In last decade, additive coupled Fibonacci sequences are popularized, but multiplicative coupled difference equations or recurrence relations are less known. In this paper we present fundamental properties of multiplicative coupled Fibonacci sequences of second order.

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1. INTRODUCTION:

The coupled difference equations or recurrence relations are popularized in last decade. They involve two sequences of integers in which the elements of one sequence are part of the generalization of the other, and vice versa. We can say that these are generalization of ordinary recursive sequences and many results can be developed for considering the two sequences are identical.

The concept of coupled Fibonacci sequence was first introduced by K. T. Atanassov [5] and also discussed many curious properties and new direction of generalization of Fibonacci sequence in [2], [3] and [6]. He was defined and studied about four different ways to generate coupled sequences and called them 2-Fibonacci sequence (or 2-F sequences). This was new direction of Fibonacci sequence generalizations.

In this paper, we present new ideas in generalization of Fibonacci sequences in the case of one or more sequences. We describe basic concepts that will be used to construct multiplicative coupled Fibonacci sequences of second order. Further, we shall describe fundamental properties.

2. MULTIPLICATIVE FIBONACCI SEQUENCE:

An interesting variation on the Fibonacci sequence is found that a new term in sequence is obtained by multiplying the previous two terms. P. Glaister [7] defined the Multiplicative Fibonacci sequence by

$$F_{n+1} = F_n F_{n-1} \text{ for n} \geq 0 \text{ and } F_0 = 1, F_1 = 2 \tag{2.1}$$

The few terms of the sequence is 1, 2, 2, 4, 8, 32, 256… which is same as a sequence of power of two and indices are conventional Fibonacci numbers.

The recurrence relation (2.1) can be written as

$$F_{n+1} = 2^{F_{n-2}} \text{ for n} \geq 1 \text{ and } F_{-1} = 1, F_0 = 1. \tag{2.2}$$

Multiplicative Fibonacci sequence generalized by P. Hope [8] as

$$x_{n+2} = x_{n+1} x_n, \text{ for n} \geq 0 \text{ and } \mathrm{x}_0 = a, x_1 = b, \tag{2.3}$$

where a and b are real numbers.

It can be written as

$$x_n = a^{F_{n-1}} b^{F_n} \text{ for n} \geq 1. \tag{2.4}$$

These are case of Fibonacci words [1]. Multiplicative pattern can be used for Fibonacci sequence in the case of more sequences.

3. MULTIPLICATIVE COUPLED FIBONACCI SEQUENCE:

K. T. Atanassov [4] and [6] notifies four different schemes in multiplicative form for coupled Fibonacci sequences.

Let $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ be two infinite sequences and four arbitrary real numbers a, b, c and d be given. The four different multiplicative schemes for 2- Fibonacci sequences are as follows:

First Scheme:

$$\alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d$$
$$\alpha_{n+2} = \beta_{n+1}.\beta_n, \quad n \geq 0 \tag{3.1}$$
$$\beta_{n+2} = \alpha_{n+1}.\alpha_n, \quad n \geq 0.$$

Second Scheme:

$$\alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d$$
$$\alpha_{n+2} = \alpha_{n+1}.\beta_n, \quad n \geq 0 \tag{3.2}$$
$$\beta_{n+2} = \beta_{n+1}.\alpha_n, \quad n \geq 0.$$

Third Scheme:

$$\alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d$$
$$\alpha_{n+2} = \beta_{n+1}.\alpha_n, \quad n \geq 0 \tag{3.2}$$
$$\beta_{n+2} = \alpha_{n+1}.\beta_n, \quad n \geq 0.$$

Fourth Scheme:

$$\alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d$$
$$\alpha_{n+2} = \alpha_{n+1}.\alpha_n, \quad n \geq 0. \tag{3.4}$$
$$\beta_{n+2} = \beta_{n+1}.\beta_n, \quad n \geq 0$$

First few terms of first scheme (3.1) are as under:

n	α_n	β_n
0	a	b
1	c	d
2	bd	ac
3	acd	bcd
4	abc^2d	$abcd^2$
5	$ab^2c^2d^3$	$a^2bc^3d^2$
6	$a^3b^2c^4d^4$	$a^2b^3c^4d^4$
7	$a^4b^4c^7d^6$	$a^4b^4c^6d^7$
8	$a^6b^7c^{10}d^{11}$	$a^7b^6c^{11}d^{10}$
9	$a^{11}b^{10}c^{17}d^{17}$	$a^{10}b^{11}c^{17}d^{17}$

Now we present fundamental properties of scheme (3.1).

Theorem 3.1. For every integer $n \geq 0$:

(a) $\beta_0.\alpha_{3n+3} = \alpha_0.\beta_{3n+3}$,

$(b)\ \beta_1.\alpha_{3n+4} = \alpha_1.\beta_{3n+4},$

$(c)\ \beta_2.\alpha_{3n+5} = \alpha_2.\beta_{3n+5}.$

Proof: (a) To prove this, we shall use induction method.
If n=0 then

$$\beta_0.\alpha_3 = \beta_0.\beta_2.\beta_1 \qquad \text{(by scheme 3.1)}$$

$$= \beta_0.\beta_1.\alpha_1.\alpha_0 \qquad \text{(by scheme 3.1)}$$

$$= \alpha_2.\alpha_1.\alpha_0 \qquad \text{(by scheme 3.1)}$$

$$= \beta_3.\alpha_0 \qquad \text{(by scheme 3.1)}$$

Thus the result is true for n= 0.

Let us assume that the result is true for some integer $n \geq 1.$ then

$$\beta_0.\alpha_{3n+6} = \beta_0.\alpha_{3n+6} \qquad \text{(by scheme 3.1)}$$

$$= \beta_0.\beta_{3n+5}.\beta_{3n+4} \qquad \text{(by scheme 3.1)}$$

$$= \beta_0.(\alpha_{3n+4}.\alpha_{3n+3}).\beta_{3n+4} \qquad \text{(by scheme 3.1)}$$

$$= \alpha_{3n+4}.(\alpha_0\beta_{3n+3}).\beta_{3n+4} \qquad \text{(by induction hypothesis)}$$

$$= \alpha_0.\alpha_{3n+4}.\alpha_{3n+5} \qquad \text{(by scheme 3.1)}$$

$$= \alpha_0.\beta_{3n+6} \qquad \text{(by scheme 3.1)}$$

Hence the result is true for all integers $n \geq 0$.

Similar proofs can be given for remaining parts (b) and (c).

Theorem 3.2. For every integer $n \geq 0$:

$$(a)\ \alpha_{3n+3} = \beta_1.\prod_{i=0}^{3n+1}\alpha_i,$$

$$(b)\ \beta_{3n+3} = \alpha_1.\prod_{i=0}^{3n+1}\beta_i,$$

$$(c)\ \alpha_{3n+4} = \alpha_1.\prod_{i=0}^{3n+2}\alpha_i,$$

$$(d)\ \beta_{3n+4} = \beta_1.\prod_{i=0}^{3n+2}\beta_i,$$

$$(e)\ \alpha_{3n+5} = \frac{\alpha_1}{\alpha_0}.\prod_{i=0}^{3n+3}\alpha_i,$$

$$(f)\ \beta_{3n+5} = \frac{\beta_1}{\beta_0}.\prod_{i=0}^{3n+3}\beta_i.$$

Theorem 3.3. For every integer $n \geq 0$:

$$\alpha_n.\beta_n = \alpha_0^{F_{n-1}}.\beta_0^{F_{n-1}}.\alpha_1^{F_n}.\beta_1^{F_n}.$$

Theorem 3.4. For every integer $n \geq 0$:

$$\text{(a)}\ \frac{\alpha_{3n+7}}{\alpha_{3n+4}} = \alpha_0^{F_{3n+4}}.\beta_0^{F_{3n+4}}.\alpha_1^{F_{3n+5}}.\beta_1^{F_{3n+5}},$$

$$\text{(b)}\ \frac{\beta_{3n+7}}{\beta_{3n+4}} = \alpha_0^{F_{3n+4}}.\beta_0^{F_{3n+4}}.\alpha_1^{F_{3n+5}}.\beta_1^{F_{3n+5}},$$

(c) $\dfrac{\alpha_{3n+6}}{\alpha_{3n+3}} = \alpha_0^{F_{3n+3}}.\beta_0^{F_{3n+3}}.\alpha_1^{F_{3n+4}}.\beta_1^{F_{3n+4}},$

(d) $\dfrac{\beta_{3n+6}}{\beta_{3n+3}} = \alpha_0^{F_{3n+3}}.\beta_0^{F_{3n+3}}.\alpha_1^{F_{3n+4}}.\beta_1^{F_{3n+4}},$

(e) $\dfrac{\alpha_{3n+5}}{\alpha_{3n+2}} = \alpha_0^{F_{3n+2}}.\beta_0^{F_{3n+2}}.\alpha_1^{F_{3n+3}}.\beta_1^{F_{3n+3}},$

(f) $\dfrac{\beta_{3n+5}}{\beta_{3n+2}} = \alpha_0^{F_{3n+2}}.\beta_0^{F_{3n+2}}.\alpha_1^{F_{3n+3}}.\beta_1^{F_{3n+3}}.$

Theorem 3.5 : For every integer $n \geq 0$:

$\alpha_{n+4} = \alpha_{n+2}.\alpha_{n+1}^2.\alpha_n,$

$\beta_{n+4} = \beta_{n+2}.\beta_{n+1}^2.\beta_n.$

Theorem 3.6 : For every integer $n \geq 0$:

(a) $\alpha_n.\alpha_{n+1}.\alpha_{n+2} = (\alpha_0.\beta_0)^{F_{n+1}}(\alpha_1.\beta_1)^{F_{n+2}},$

(b) $\beta_n.\beta_{n+1}.\beta_{n+2} = (\alpha_0.\beta_0)^{F_{n+1}}(\alpha_1.\beta_1)^{F_{n+2}}.$

Theorem 3.7 : For every integer $n \geq 0$:

(a) $\dfrac{\alpha_{n+3}}{\alpha_n} = (\alpha_0.\beta_0)^{F_n}(\alpha_1.\beta_1)^{F_{n+1}},$

(b) $\dfrac{\beta_{n+3}}{\beta_n} = (\alpha_0.\beta_0)^{F_n}(\alpha_1.\beta_1)^{F_{n+1}}.$

The above fundamental properties can be proved by induction method easily.

K. T. Atanassov [6] was defined explicit formula for α_n *and* β_n.

Theorem 3.8 : For every integer $n \geq 0$:

$$\alpha_{n+2} = \alpha_0^{\frac{1}{2}\left(F_{n+1}+3.\left[\frac{n+2}{3}\right]-n-1\right)} . \beta_0^{\frac{1}{2}\left(F_{n+1}-3.\left[\frac{n+2}{3}\right]+n+1\right)} . \alpha_1^{\frac{1}{2}\left(F_{n+2}-3.\left[\frac{n}{3}\right]+n-1\right)} . \beta_1^{\frac{1}{2}\left(F_{n+2}+3.\left[\frac{n}{3}\right]-n+1\right)F_{3n+5}},$$

$$\beta_{n+2} = \alpha_0^{\frac{1}{2}\left(F_{n+1}-3.\left[\frac{n+2}{3}\right]+n+1\right)} . \beta_0^{\frac{1}{2}\left(F_{n+1}+3.\left[\frac{n+2}{3}\right]-n-1\right)} . \alpha_1^{\frac{1}{2}\left(F_{n+2}+3.\left[\frac{n}{3}\right]-n+1\right)} . \beta_1^{\frac{1}{2}\left(F_{n+2}+3.\left[\frac{n}{3}\right]+n-1\right)}.$$

where [ ] denote for greatest integer function.

4. CONCLUSION

This paper describes multiplicative coupled Fibonacci sequence of second order. We have established the various results of coupled Fibonacci sequence that shows relation between two sequences as well as in conventional Fibonacci sequence. Similar results can be developed for remaining scheme of multiplicative form. The idea can be implemented over other recursive sequence of second order in two or more sequences.

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