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S-H Fuzzy Partition and Fuzzy Equivalence Relation

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Abstract

In this paper we will introduce a certain collection of fuzzy subsets of a nonempty set X. This certain collection gives raise to a useful new fuzzy partition. This new fuzzy partition gives us a way of generating a fuzzy equivalence relation as in the ordinary case; any partition generates an equivalence relation.

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1 Introduction

The fuzzy set theory was introduced by Zedeh (1965) in [8] in order to provide a scheme for handling a variety of problems in which a fundamental role is played by an indefiniteneis arising from a sort of intrinsic ambiguity. As fuzzy equivalence classes and fuzzy partitions plays a major role in many topics in fuzzy theory, such as fuzzy measure, fuzzy integration and fuzzy algebra, see [4] and [5].

In [2], the authors proposed a new definition of a fuzzy equivalence class such that it has built a certain partition which has been discussed in reference [4]. In the ordinary case we know that each partition T of X gives rise to an equivalence relation R defined by $xRy \Leftrightarrow x, y \in P$ for some $P \in T$. So, this leads us to the following question : Can we define a way to obtain the fuzzy eqivalence relation that was introduced in [7] from the fuzzy partition that was introduced in [4]? The answer of this question is that when the definition of the fuzzy partition was designed, it wasn't taken care of it to be fitting with the definition of the fuzzy equivalence relation. So, in this article, we will introduce a new definition of a fuzzy partition which is the same partition introduced in [4] and [2] containing an additional condition. For the importance of this partition, we will call it the *Shakhatreh-Hayajneh* (*S-H fuzzy partition*). This additional condition gives us the ability to generate a fuzzy equivalence relation.

In [1], the authors introduce some new properties of the set of all fuzzy equivalence classes $\widetilde{[X]}$ which were introduced in [2]. In this article, the authors were amazed by noticing that the set of all fuzzy equivalence classes $\widetilde{[X]}$ form the S-H fuzzy partition of X. This fact gives us a feeling of the suitability of the definition of the S-H fuzzy partition. Throughout the proof of this fact, we will use a useful property of the set of all fuzzy equivalence classes $\widetilde{[X]}$ introduced in [1] which says that $\widetilde{[x]} \cap \widetilde{[y]} \leq \widetilde{[x]}(y)$.

2 Definition and preliminaries

Let X be a universal set and P a subset of X. Then the characteristic function χ_P of the set P is defined to be the function $\chi_P : X \to [0,1]$ given by

$$\chi_P(x) = \begin{cases} 1 & \text{if } x \in P \ , \\ 0 & \text{if } x \notin P \ . \end{cases}$$

Throughout this paper, one can think of the fuzzy set λ in X as a function from X to $[0,1]$. Sometimes one can keep the symbol λ for the fuzzy set and associate it with a function μ_λ from X to $[0,1]$ called the membership function. We can also write $\lambda = (x, \mu_\lambda(x))$. The fuzzy set λ is said to be contained, $\subseteq$, in the fuzzy set μ if and only if $\lambda(x) \leq \mu(x)$ for each $x \in X$. Let $A := \{\lambda_\gamma : \gamma \in \Gamma\}$ be a collection of fuzzy sets. Then the union and the intersection are defined as follows: $(\bigcup_{\gamma\in\Gamma})(x) = sup_{\gamma\in\Gamma}(x)$ and $(\bigcap_{\gamma\in\Gamma})(x) = inf_{\gamma\in\Gamma}(x)$. Let $T := \{P_\gamma : \gamma \in \Gamma\}$ be a collection of subsets of X. Then T is called an ordinary partition of X if and only if $P_\gamma \neq \phi$ for each $\gamma \in \Gamma$, $P_\gamma \cap P_\eta = \phi$ whenever $\gamma \neq \eta$ and $X = \bigcup_{\gamma\in\Gamma} P_\gamma$.

Now, we will state some definitions and theorems which were given in [7], [4], [6], and [2].

Definition 2.1 *(Weak-Separated Fuzzy Subsets, see [6]) Let $\widetilde{F}$ be a collection of fuzzy subsets of a nonempty set X and $\widetilde{A}, \widetilde{B} \in \widetilde{F}$ with $\widetilde{A} \neq \widetilde{B}$.*

If $\mu_{\widetilde{A}\cap\widetilde{B}}(x) < 0.5$, $\forall x \in X$, then $\widetilde{A}$ and $\widetilde{B}$ are called Weak-Separated fuzzy subsets.

Definition 2.2 *(Fuzzy Equivalence Relation, see [7]) Let $\widetilde{R}$ be a fuzzy relation on a nonempty set X.*

(a) *$\widetilde{R}$ is reflexive on X iff $\mu_{\widetilde{R}}(x,x) = 1$, $\forall (x,x) \in X \times X$.*

(b) *$\widetilde{R}$ is symmetric on X iff $\mu_{\widetilde{R}}(x,y) = \mu_{\widetilde{R}}(y,x)$, $\forall (x,y),(y,x) \in X \times X$.*

(c) *$\widetilde{R}$ is transitive on X iff $\mu_{\widetilde{R}}(x,z) \geq max_y \left\{min\left\{\mu_{\widetilde{R}}(x,y), \mu_{\widetilde{R}}(y,z)\right\}\right\}$, $\forall (x,z),(x,y),(y,z) \in X \times X$.*

Definition 2.3 *(Fuzzy Partition, see [4]) Let X be a nonempty set. A fuzzy partition $\widetilde{T}$ of X is a set of nonempty fuzzy subsets of X such that*

(a) *If* $\widetilde{A}, \widetilde{B} \in \widetilde{T}$ *and* $\widetilde{A} \neq \widetilde{B}$, *then* $(\widetilde{A} \cap \widetilde{B})(x) < 0.5$.

(b) $\bigcup_{\widetilde{w} \in \widetilde{T}} \widetilde{w} = X$.

Definition 2.4 *(Elements which have Strong Bond with* x*, see [2]) Let* $\widetilde{R}$ *be a fuzzy relation on a nonempty set* X *and* $x \in X$*. Then*

$B(x) = \{y \in X : \mu_{\widetilde{R}}(y, x) \geq 0.5\}$

is called the set of all elements which has strong bond with x*.*

Definition 2.5 *(Fuzzy Equivalence Classes, see [1]) Let* $\widetilde{R}$ *be a fuzzy equivalence relation on a nonempty set* X *and* $x \in X$*. Then*

(a) $\widetilde{[x]} = \left\{(y, \mu_{\widetilde{[x]}}(y)) : y \in X\right\}$, *where*

$$\mu_{\widetilde{[x]}}(y) = \begin{cases} 1 & \text{if } y \in B(x) \ , \\ \mu_{\widetilde{R}}(x, y) & \text{if } y \notin B(x) \ . \end{cases}$$

is called the fuzzy equivalence class determined by x *.*

(b) $\widetilde{[X]} = \left\{\widetilde{[x]} : x \in X\right\}$

is called the set of all fuzzy equivalence classes.

Definition 2.6 *(Weakly Empty Fuzzy Subset, see [4]) Let* X *be a nonempty set and* $\widetilde{A}$ *be a fuzzy subset of* X*. Then we call* $\widetilde{A}$ *weakly empty fuzzy subset of* X *if* $\mu_{\widetilde{A}}(x) < 0.5 \ \forall \ x \in X$.

Theorem 2.1 *(see [2]) Let* $\widetilde{R}$ *be a fuzzy equivalence relation on a nonempty set* X *and* $\widetilde{[X]}$ *be the set of all fuzzy equivalence classes. Then*

(a) *If* $x \neq y$, *then* $\widetilde{[x]} \cap \widetilde{[y]} < 0.5$.

(b) $\bigcup_{x \in X} \widetilde{[x]} = X$.

Theorem 2.2 *(see [1]) Let* $\widetilde{R}$ *be a fuzzy equivalence relation on a nonempty set* X *and* $\widetilde{[X]}$ *be the set of all fuzzy equivalence classes. Then* $\mu_{\widetilde{[x]} \cap \widetilde{[z]}}(y) \leq \mu_{\widetilde{[z]}}(x) \ \forall \ x, y, z \in X$.

Remark 2.1 *For simplicity, we can use the notation* $\widetilde{A}(x)$ *instead of* $\mu_{\widetilde{A}}(x)$ *to denote the degree of* x *in the fuzzy set* $\widetilde{A}$ *on* X*. By* $\widetilde{A} \leq \widetilde{B}$, *we mean* $\mu_{\widetilde{A}}(x) \leq \mu_{\widetilde{B}}(x) \ \forall \ x \in X$.

3 S-H Fuzzy Partition

Now, we will introduce a new definition of a fuzzy partition which contains an additional condition. This additional condition gives us the ability to generate a fuzzy equivalence relation. For the importance of this condition, we will call it the *S-H condition* and we will call this new partition the *S-H fuzzy partition*.

Definition 3.1 *(S-H Fuzzy Collection) Let $\widetilde{F}$ be a collection of fuzzy subsets of a nonempty set X. $\widetilde{F}$ is called S-H collection if and only if $\widetilde{B} \cap \widetilde{A} \leq \widetilde{B}(a)$ whenever $\widetilde{A}, \widetilde{B} \in \widetilde{F}$ such that $\widetilde{A}(a) = 1$.*

Example 3.1 *Let $T := \{P_\gamma : \gamma \in \Gamma\}$ form an ordinary partition. Then $\widetilde{T} := \left\{\chi_{P_\gamma} : \gamma \in \Gamma\right\}$ forms an S-H fuzzy collection.*

Proof Let $\gamma, \eta \in \Gamma$ and $a \in X$ such that χ_{P_γ} is a non weakly empty fuzzy subset and $\chi_{P_\eta}(a) = 1$. Now if $P_\gamma = P_\eta$, then we have $\chi_{P_\gamma}(a) = 1$. This implies that $\chi_{P_\gamma} \cap \chi_{P_\eta} \leq \chi_{P_\gamma}(a)$ and we are done. Also if $P_\gamma \neq P_\eta$, then we have $\chi_{P_\gamma} \cap \chi_{P_\eta} = 0$, which implies that $\chi_{P_\gamma} \cap \chi_{P_\eta} \leq \chi_{P_\gamma}(a)$ and we are done. By above discussion $\widetilde{T}$ forms an S-H fuzzy collection. □

Example 3.2 *Let $X = \{x_0, x_1, x_2, x_3, x_4\}$ and let $\widetilde{F} = \left\{\widetilde{A_1}, \widetilde{A_2}, \widetilde{A_3}, \widetilde{A_4}, \widetilde{A_5}\right\}$ be a collection of fuzzy subsets of X, where*

$\widetilde{A_1} = \{(x_0, 1.0), (x_1, 0.9), (x_2, 0.8), (x_3, 0.7), (x_4, 0.1)\}$
$\widetilde{A_2} = \{(x_0, 0.9), (x_1, 1.0), (x_2, 0.8), (x_3, 0.7), (x_4, 0.1)\}$
$\widetilde{A_3} = \{(x_0, 0.8), (x_1, 0.8), (x_2, 1.0), (x_3, 0.7), (x_4, 0.1)\}$
$\widetilde{A_4} = \{(x_0, 0.7), (x_1, 0.7), (x_2, 0.7), (x_3, 1.0), (x_4, 0.1)\}$
$\widetilde{A_5} = \{(x_0, 0.1), (x_1, 0.1), (x_2, 0.1), (x_3, 0.1), (x_4, 0.1)\}$

Note that $\widetilde{A_i} \cap \widetilde{A_j} \leq \widetilde{A_j}(x)$, where $\widetilde{A_i}(x) = 1$ and $i \in \{1, 2, 3, 4\}$. Therefore $\widetilde{F}$ is S-H collection. We can see also that $\widetilde{A_5}$ is a weakly empty fuzzy set and

$$\bigcup \widetilde{A_i} = \{(x_0, 1), (x_1, 1), (x_2, 1), (x_3, 1), (x_4, 0.1)\} \neq X.$$

Also note that $\mu_{\widetilde{A_1} \cap \widetilde{A_2}}(x_0) = 0.9 > 0.5$.

Definition 3.2 *(The S-H Fuzzy Partition) Let X be a nonempty set. By an S-H fuzzy partition $\widetilde{T}$ of X, we mean a set of non weakly empty fuzzy subsets of X such that*

(a) *If $\widetilde{A}, \widetilde{B} \in \widetilde{T}$ and $\widetilde{A} \neq \widetilde{B}$, then $(\widetilde{A} \cap \widetilde{B})(x) < 0.5$, i.e., any two non identical fuzzy subsets are Weak-Separated fuzzy subsets.*

(b) *$\bigcup_{\widetilde{w} \in \widetilde{T}} \widetilde{w} = X$.*

(c) *$\widetilde{T}$ is S-H collection.*

Remark 3.1 *The last condition in Definition 3.2 is independent of the other conditions. In Example 3.2, we can see that Condition (c) does not imply Condition (a) or Condition (b). Also Condition (a) and Condition (b) do not imply Condition (c). To see that we need the following example.*

Example 3.3 *Let $X = \{x, y, z\}$ and let $\widetilde{F} = \left\{\widetilde{A}, \widetilde{B}, \widetilde{C}\right\}$ be a collection of fuzzy subsets of X, where*

$\widetilde{A} = \{(x, 1.0), (y, 0.1), (z, 0.4)\}$

$\widetilde{B} = \{(x, 0.1), (y, 1.0), (z, 0.4)\}$

$\widetilde{C} = \{(x, 0.4), (y, 0.4), (z, 1.0)\}$

Note that $(\widetilde{A} \cap \widetilde{B})(z) = 0.4 > \widetilde{B}(x) = 0.1$. Therefore, Condition (c) does not hold, i.e., $\widetilde{F}$ is not an S-H collection. We can see also that $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ are non weakly empty fuzzy sets and $\bigcup \widetilde{A_i} = X$. Note also that $\widetilde{A} \cap \widetilde{B} < 0.5$, $\widetilde{A} \cap \widetilde{C} < 0.5$ and $\widetilde{B} \cap \widetilde{C} < 0.5$.

The next lemma contains some hidden properties of the fuzzy partition introduced in [4] and [2]. Part (a) of this lemma will be used in proving that the fuzzy equivalence relation introduced in Definition 6.1 below is a well defined relation. It is also used in proving the next theorems.

Lemma 3.1 *Let $\widetilde{T}$ be S-H fuzzy partition of a nonempty set X.*

(a) *If $x \in X$, then there exist a unique $\widetilde{A} \in \widetilde{T}$ such that $\mu_{\widetilde{A}}(x) = 1$.*

(b) *If $x \in X$, $\widetilde{A} \in \widetilde{T}$ such that $\mu_{\widetilde{A}}(x) \geq 0.5$, then $\mu_{\widetilde{A}}(x) = 1$.*

Proof

(a)

- (Existence) Suppose that $\mu_{\widetilde{A}}(x) \neq 1 \; \forall \, \widetilde{A} \in \widetilde{T}$. Then $\max \left\{\mu_{\widetilde{A}}(x) : \; \widetilde{A} \in \widetilde{T}\right\} \neq 1$. This implies that $\mu_{\bigcup_{\widetilde{A} \in \widetilde{T}} \widetilde{A}}(x) \neq 1$ which contradicts part(b) of Definition 3.2.
- (Uniqueness) Suppose that $\exists \; \widetilde{A}, \widetilde{B} \in \widetilde{T}$ such that $\widetilde{A} \neq \widetilde{B}$ and $\mu_{\widetilde{A}}(x) = \mu_{\widetilde{B}}(x) = 1$. Then $\min\left\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)\right\} = 1 \geq 0.5$. This implies that $\mu_{\widetilde{A} \cap \widetilde{B}}(x) \geq 0.5$ which contradicts part (a) of Definition 3.2.

(b) Suppose on contrary that $\mu_{\widetilde{A}}(x) \neq 1$. But by assumption we have $\mu_{\widetilde{A}}(x) \geq 0.5$. Then there exist a unique $\widetilde{B} \in \widetilde{T}$ such that $\mu_{\widetilde{B}}(x) = 1$ and $\widetilde{B} \neq \widetilde{A}$ which implies that $\min\left\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)\right\} \geq 0.5$. Thus $\mu_{\widetilde{A} \cap \widetilde{B}}(x) \geq 0.5$ which contradicts part (a) of Definition 3.2 and so $\mu_{\widetilde{A}}(x) = 1$. □

Now, the remaining sections give us a feeling of the suitability of the definition of the S-H fuzzy partition.

4 S-H Fuzzy Partition Covers The Ordinary Case

Theorem 4.1 *Let $T := \{P_\gamma : \gamma \in \Gamma\}$ forms an ordinary partition. Then $\widetilde{T} := \left\{\chi_{P_\gamma} : \gamma \in \Gamma\right\}$ forms an S-H fuzzy partition.*

Proof Firstly, by Example 3.1, $\widetilde{T}$ forms an S-H fuzzy collection.

If T forms an ordinary partition, then $P_\gamma \neq \phi$. Therefore, $\exists\, x \in X$ such that $x \in P_\gamma$. This implies that $\exists\, x \in X$ such that $\chi_{P_\gamma}(x) = 1$, i.e., χ_{P_γ} is not a weakly empty fuzzy set.

Also $\forall\ y \in X, \exists\ \gamma \in \Gamma$ such that $\chi_{P_\gamma}(y) = 1$, which implies that $\bigcup_{i\in\Gamma} \chi_{P_i}(y) = \chi_{P_\gamma}(y) = 1$, i.e., $\bigcup_{i\in\Gamma} \chi_{P_i} = X$.

Now if $P_\gamma \neq P_\eta$, then we claim that $P_\gamma \cap P_\eta$ is a weakly empty fuzzy set, i.e., $P_\gamma \cap P_\eta < 0.5$ (Weak-Separated fuzzy subsets). To prove the claim we have two cases. If $y \notin P_\gamma$, then $\chi_{P_\gamma}(y) = 0$. And so $\min\left\{\chi_{P_\gamma}(y), \chi_{P_\eta}(y)\right\} \leq 0.5\ \forall\ y \in X$. If $y \in P_\gamma$, then $y \notin P_\eta$. Therefore, $\chi_{P_\eta}(y) = 0$. And so $\min\left\{\chi_{P_\gamma}(y), \chi_{P_\eta}(y)\right\} \leq 0.5\ \forall\ y \in X$. □

The next theorem gives a wealthy collection of S-H fuzzy partitions.

5 A Way of Generating S-H Fuzzy Partition

The authors noticed that the set of all fuzzy equivalence classes $\widetilde{[X]}$ introduced in [1], forms an S-H fuzzy partition of X. So we are catching a practical way to construct an S-H fuzzy partition. To construct an S-H fuzzy partition, it is just to search for any fuzzy equivalence relation and to compute the set of all fuzzy equivalence classes $\widetilde{[X]}$.

Theorem 5.1 *Let $\widetilde{R}$ be a fuzzy equivalence relation on a nonempty set X, then $\widetilde{[X]}$ is S-H fuzzy partition of X.*

Proof Firstly, it is known that $\widetilde{[x]}$ is a non weakly empty since $\mu_{\widetilde{[x]}}(x) = 1$. Thus $\widetilde{[X]}$ is a collection of non weakly empty fuzzy subsets. (a) and (b) were proved in [2]. (c) follows directly from Theorem 2.2. □

Example 5.1 *Let $X = \{x_0, x_1, x_2, x_3\}$ and let $\widetilde{R}$ be the fuzzy relation in $X \times X$ given in Table 1. We can show that $\widetilde{R}$ is a fuzzy equivalence relation.*

	x_0	x_1	x_2	x_3
x_0	*1*	*0.7*	*0.3*	*0.1*
x_1	*0.7*	*1*	*0.3*	*0.1*
x_2	*0.3*	*0.3*	*1*	*0.1*
x_3	*0.1*	*0.1*	*0.1*	*1*

Table 1.

Now to construct an S-H fuzzy partition, we are to compute the set of all fuzzy equivalence classes $\widetilde{[X]}$ *as follows*

$\widetilde{[x_0]} = \{(x_0, 1.0), (x_1, 1.0), (x_2, 0.3), (x_3, 0.1)\}$
$\widetilde{[x_1]} = \{(x_0, 1.0), (x_1, 1.0), (x_2, 0.3), (x_3, 0.1)\}$
$\widetilde{[x_2]} = \{(x_0, 0.3), (x_1, 0.3), (x_2, 1.0), (x_3, 0.1)\}$
$\widetilde{[x_3]} = \{(x_0, 0.1), (x_1, 0.1), (x_2, 0.1), (x_3, 1.0)\}$

Finally, by setting $\widetilde{T} = \widetilde{[X]}$, *we have an S-H fuzzy partition.*

6 How to Generate Fuzzy Equivalence Relation Using an S-H Fuzzy Partition?

The next definition demonstrates a way of generating a fuzzy equivalence relation from the S-H fuzzy partition. We will prove that this relation is an equivalence relation and then when we start with any S-H fuzzy partition, we can get a fuzzy equivalence relation.

Definition 6.1 *Let* $\widetilde{T}$ *be an S-H fuzzy partition of a nonempty set* X. *We define a fuzzy relation* $X/\widetilde{T}$ *on* X *by the following,*

$\mu_{X/\widetilde{T}}(x, y) = \mu_{\widetilde{A}}(y)$ *where* $\widetilde{A} \in \widetilde{T}$ *such that* $\mu_{\widetilde{A}}(x) = 1$.

Remark 6.1 *We can see that the above definition is well defined by using Part(a) of Lemma 3.1.*

Theorem 6.1 *Let* $\widetilde{T}$ *be an S-H fuzzy partition of a nonempty set* X. *Then the fuzzy relation* $X/\widetilde{T}$ *is a fuzzy equivalence relation on* X.

Proof

(a) ($X/\widetilde{T}$ is reflexive) $\mu_{X/\widetilde{T}}(x, x) = \mu_{\widetilde{A}}(x)$ where $\widetilde{A} \in \widetilde{T}$ and $\mu_{\widetilde{A}}(x) = 1$ by Definition 6.1 which implies that $\mu_{X/\widetilde{T}}(x, x) = 1$, and so $X/\widetilde{T}$ is reflexive on X.

(b) ($X/\widetilde{T}$ is symmetric) Let $d, e \in X$, then by Lemma 3.1, $\exists\ \widetilde{D}, \widetilde{E} \in \widetilde{T}$ such that $\mu_{\widetilde{D}}(d) = \mu_{\widetilde{E}}(e) = 1$. Now using part(c) of Definition 3.2 and taking $\widetilde{A} = \widetilde{D}$ and $\widetilde{B} = \widetilde{E}$, we have $\mu_{\widetilde{D} \cap \widetilde{E}}(e) \leq \mu_{\widetilde{E}}(d)$. But $\mu_{\widetilde{D} \cap \widetilde{E}}(e) = \min\{\mu_{\widetilde{D}}(e), \mu_{\widetilde{E}}(e)\} = \min\{\mu_{\widetilde{D}}(e), 1\} = \mu_{\widetilde{D}}(e)$. And so $\mu_{\widetilde{D}}(e) \leq \mu_{\widetilde{E}}(d)$. In the same way and by taking $\widetilde{A} = \widetilde{E}$ and $\widetilde{B} = \widetilde{D}$, we have $\mu_{\widetilde{E}}(d) \leq \mu_{\widetilde{D}}(e)$ which implies that $\mu_{\widetilde{E}}(d) = \mu_{\widetilde{D}}(e)$. But by Definition 6.1, we have $\mu_{\widetilde{E}}(d) = \mu_{X/\widetilde{T}}(e, d)$ and $\mu_{\widetilde{D}}(e) = \mu_{X/\widetilde{T}}(d, e)$ and so $\mu_{X/\widetilde{T}}(e, d) = \mu_{X/\widetilde{T}}(d, e)$. Thus $X/\widetilde{T}$ is symmetric on X.

(c) ($X/\widetilde{T}$ is transitive) Let $d, e, f \in X$, then by Lemma 3.1 $\exists\ \widetilde{D}, \widetilde{E}, \widetilde{F} \in \widetilde{T}$ such that $\mu_{\widetilde{D}}(d) = \mu_{\widetilde{E}}(e) = \mu_{\widetilde{F}}(f) = 1$. Now using part(c) of Definition 3.2 and taking $\widetilde{A} = \widetilde{D}$ and $\widetilde{C} = \widetilde{E}$, we have $\mu_{\widetilde{D} \cap \widetilde{E}}(f) \leq \mu_{\widetilde{E}}(d)$. Thus $\min\{\mu_{\widetilde{D}}(f), \mu_{\widetilde{E}}(f)\} \leq \mu_{\widetilde{E}}(d)$. But by part(b) of this theorem, $\mu_{\widetilde{E}}(f) = \mu_{\widetilde{F}}(e)$. And so $\min\left\{\mu_{\widetilde{D}}(f), \mu_{\widetilde{F}}(e)\right\} \leq \mu_{\widetilde{E}}(d)$. Now using Definition 6.1, we have $\mu_{\widetilde{D}}(f) = \mu_{X/\widetilde{T}}(d, f)$, $\mu_{\widetilde{F}}(e) = \mu_{X/\widetilde{T}}(f, e)$ and $\mu_{\widetilde{E}}(d) = \mu_{X/\widetilde{T}}(e, d)$. And so $\min\left\{\mu_{X/\widetilde{T}}(d, f), \mu_{X/\widetilde{T}}(f, e)\right\} \leq \mu_{X/\widetilde{T}}(e, d)$. Thus $X/\widetilde{T}$ is transitive on X.

Now from (a),(b) and (c), we conclude that $X/\widetilde{T}$ is a fuzzy equivalence relation on X. □

7 Example on Generating Fuzzy Equivalence Relation

Example 7.1 *Let* $X = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ *and let* $\widetilde{T} = \left\{\widetilde{A_1}, \widetilde{A_2}, \widetilde{A_3}, \widetilde{A_4}, \widetilde{A_5}\right\}$ *be a collection of fuzzy subsets of* X*, where*

$\widetilde{A_1} =$
$\{(x_0, 1.0), (x_1, 1.0), (x_2, 1.0), (x_3, 1.0), (x_4, 1.0), (x_5, 1.0), (x_6, 0.4), (x_7, 0.3), (x_8, 0.2), (x_9, 0.1)\}$

$\widetilde{A_2} =$
$\{(x_0, 0.4), (x_1, 0.4), (x_2, 0.4), (x_3, 0.4), (x_4, 0.4), (x_5, 0.4), (x_6, 1.0), (x_7, 0.3), (x_8, 0.2), (x_9, 0.1)\}$

$\widetilde{A_3} =$
$\{(x_0, 0.3), (x_1, 0.3), (x_2, 0.3), (x_3, 0.3), (x_4, 0.3), (x_5, 0.3), (x_6, 0.3), (x_7, 1.0), (x_8, 0.2), (x_9, 0.1)\}$

$\widetilde{A_4} =$
$\{(x_0, 0.2), (x_1, 0.2), (x_2, 0.2), (x_3, 0.2), (x_4, 0.2), (x_5, 0.2), (x_6, 0.2), (x_7, 0.2), (x_8, 1.0), (x_9, 0.1)\}$

$\widetilde{A_5} =$
$\{(x_0, 0.1), (x_1, 0.1), (x_2, 0.1), (x_3, 0.1), (x_4, 0.1), (x_5, 0.1), (x_6, 0.1), (x_7, 0.1), (x_8, 0.1), (x_9, 1.0)\}$

We recognize that $\mu_{\widetilde{A_i} \cap \widetilde{A_j}}(x) < 0.5$, where $i, j \in \{1, 2, 3, 4, 5\}$ and $i \neq j$ and that $\bigcup \widetilde{A_i} = X$. We can also see that $\widetilde{A_i}$ is not weakly empty fuzzy subset $\forall\ i \in \{1, 2, 3, 4, 5\}$. Note also that $\widetilde{A_i} \cap \widetilde{A_j} \leq \widetilde{A_j}(x)$, where $\widetilde{A_i}(x) = 1$. Thus $\widetilde{T}$ is S-H collection. By the above discussion, $\widetilde{T}$ forms an S-H partition.

Now we are ready to generate a fuzzy equivalence relation by using Definition 6.1 to get the fuzzy equivalence relation $X/\widetilde{T}$.

For example, to compute $\mu_{\widetilde{R}}(x_0, x_5)$, we search for $\widetilde{A_i} \in \widetilde{T}$ such that $\widetilde{A_i}(x_0) = 1$. But this is true for $\widetilde{A_1}$, thus $\mu_{\widetilde{R}}(x_0, x_5) = \widetilde{A_1}(x_5) = 1$. In the same way we can compute $\mu_{\widetilde{R}}(x_i, x_j)\ \forall\ i, j \in \{1, 2, 3, 4, 5\}$ and hence, we can get the fuzzy equivalence relation $X/\widetilde{T}$ as in Table 2.

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
x_0	1	1	1	1	1	1	0.4	0.3	0.2	0.1
x_1	1	1	1	1	1	1	0.4	0.3	0.2	0.1
x_2	1	1	1	1	1	1	0.4	0.3	0.2	0.1
x_3	1	1	1	1	1	1	0.4	0.3	0.2	0.1
x_4	1	1	1	1	1	1	0.4	0.3	0.2	0.1
x_5	1	1	1	1	1	1	0.4	0.3	0.2	0.1
x_6	0.4	0.4	0.4	0.4	0.4	0.4	1	0.3	0.2	0.1
x_7	0.3	0.3	0.3	0.3	0.3	0.3	0.3	1	0.2	0.1
x_8	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	1	0.1
x_9	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	1

Table 2.

8 General Example on S-H Fuzzy Partition

The next theorem demonstrates a general way of generating S-H fuzzy partitions from ordinary partitions and decreasing sequences whose terms belong to the interval $[0, 0.5)$. This theorem also can be used in proving that the above example is indeed forms an S-H partition.

Theorem 8.1 *Let X be a nonempty set, $\{A_i : i \in \mathbb{N}\}$ be an ordinary partition of X and $\{\alpha_i\}_{i=0}^{\infty}$ be a decreasing sequence such that $\alpha_i \in [0, 0.5)$. If $\widetilde{T} := \left\{\widetilde{T}_k : k \in \mathbb{N}\right\}$ is a collection of fuzzy sets, given by*

$$\widetilde{T}_k(x) = \begin{cases} \alpha_k & \text{if } k > s \ , \\ 1 & \text{if } k = s \ , \\ \alpha_s & \text{if } k < s \ . \end{cases} = \begin{cases} 1 & \text{if } k = s \ , \\ \min\{\alpha_k, \alpha_s\} & \text{if } k \neq s \ . \end{cases}$$

where $s \in \mathbb{N}$ such that $x \in A_s$, then $\widetilde{T}$ is S-H fuzzy partition.

Proof

First, note that

$$\widetilde{T}_k(x) = 1 \Leftrightarrow x \in A_k. \tag{1}$$

$\widetilde{T}$ is a collection of nonempty fuzzy subsets since $\widetilde{T}_k(x) = 1 \ \forall \ x \in A_k$. Moreover $\widetilde{T}$ is a collection of non weakly empty fuzzy subsets. We are to show that whenever $i, j \in \mathbb{N}$ such that $i \neq j$, we have

$$(\widetilde{T}_i \cap \widetilde{T}_j)(x) = \begin{cases} \alpha_s & \text{if } i < j \leq s \text{ or } j < i \leq s \ , \\ \alpha_j & \text{if } i < j \text{ and } j > s, \\ \alpha_i & \text{if } i > j \text{ and } i > s. \end{cases} \tag{2}$$

where $s \in \mathbb{N}$ such that $x \in A_s$.

Case 1 $(i < j \leq s$ or $j < i \leq s)$

Here we have also two cases.

Case (a) $(j = s$ or $i = s)$

Suppose that $j = s$. Now by the definition of $\widetilde{T}_k$, we have $\widetilde{T}_j(x) = 1$. But $i < j$, so $i < s$. This implies that $\widetilde{T}_i(x) = \alpha_s$. Also $(\widetilde{T}_i \cap \widetilde{T}_j)(x) = min\left\{\widetilde{T}_i(x), \widetilde{T}_j(x)\right\}$. Thus $(\widetilde{T}_i \cap \widetilde{T}_j)(x) = \alpha_s$. Similarly when $i = s$.

Case (b) $(j < s$ or $i < s)$

Using the definition of $\widetilde{T}_k$, we have $\widetilde{T}_i(x) = \widetilde{T}_j(x) = \alpha_s$. But $(\widetilde{T}_i \cap \widetilde{T}_j)(x) = min\left\{\widetilde{T}_i(x), \widetilde{T}_j(x)\right\}$. And $(\widetilde{T}_i \cap \widetilde{T}_j)(x) = \alpha_s$.

Case 2 $(j > i$ and $j > s)$

Using the definition of $\widetilde{T}_k$, we have $\widetilde{T}_j(x) = \alpha_j$. Also we have $\widetilde{T}_i(x) = 1$, α_s or α_i. By using the fact that $\{\alpha_i\}_{i=0}^{\infty}$ is a decreasing sequence, we have $\alpha_i \geq \alpha_j$ and $\alpha_s \geq \alpha_j$. But $(\widetilde{T}_i \cap \widetilde{T}_j)(x) = min\left\{\widetilde{T}_i(x), \widetilde{T}_j(x)\right\}$, so $(\widetilde{T}_i \cap \widetilde{T}_j)(x) = \alpha_j$.

Case 3 $(i > j$ and $i > s)$

Similarly to Case 2.

Now if $\widetilde{T}_n(y) = 1$, then by (1) we have $y \in A_n$. So for any $m \in \mathbb{N}$ we have

$$\widetilde{T}_m(y) = \begin{cases} \alpha_m & \text{if } n < m \ , \\ 1 & \text{if } m = n \ , \\ \alpha_n & \text{if } n > m \ . \end{cases} \tag{3}$$

Now we want to prove that $\widetilde{T}_n \cap \widetilde{T}_m \leq \widetilde{T}_m(y)$.

Case 1 $(n = m)$

We have $\widetilde{T}_n \cap \widetilde{T}_m = \widetilde{T}_n$, $\widetilde{T}_n \leq 1$ and $\widetilde{T}_n(y) = \widetilde{T}_m(y)$. But $\widetilde{T}_n(y) = 1$, and so $\widetilde{T}_n \cap \widetilde{T}_m \leq \widetilde{T}_m(y)$.

Case 2 $(n > m)$

Using (2), we have $\widetilde{T}_n \cap \widetilde{T}_m = \alpha_n$ or α_s, where $m < n \leq s$. But by (3), $\widetilde{T}_m(y) = \alpha_n$. Now using the fact that $\{\alpha_i\}_{i=0}^{\infty}$ is a decreasing sequence, we have $\widetilde{T}_n \cap \widetilde{T}_m \leq \widetilde{T}_m(y)$.

Case 3 $(n < m)$

Using (2), we have $\widetilde{T}_n \cap \widetilde{T}_m = \alpha_m$ or α_s, where $n < m \leq s$. But by (3), $\widetilde{T}_m(y) = \alpha_m$. Now using the fact that $\{\alpha_i\}_{i=0}^{\infty}$ is a decreasing sequence, we have $\widetilde{T}_n \cap \widetilde{T}_m \leq \widetilde{T}_m(y)$.

So $\widetilde{T}$ is an S-H fuzzy collection. Now from (2) and that $\alpha_i \in [0, 0.5)$, we can conclude that whenever $i, j \in \mathbb{N}$ such that $i \neq j$, we have $\widetilde{T_i} \cap \widetilde{T_j} < 0.5$. Note also that $(\bigcup_{k=1}^{\infty} \widetilde{T_k})(x) = sup_k \left\{\widetilde{T_k}(x)\right\} = \widetilde{T_s}(x) = 1$ where $s \in N$ such that $x \in A_s$, i.e, $\bigcup_{k=1}^{\infty} \widetilde{T_k} = X$. So $\widetilde{T}$ is an S-H fuzzy partition. □

Remark 8.1 *One can generate the fuzzy equivalence relation, $X/\widetilde{T}$, which is associated with the above S-H fuzzy partition, $\widetilde{T}$, using the Definition 6.1 as follows:*

$$\begin{aligned}
\mu_{X/\widetilde{T}}(x,y) = \widetilde{T_k}(y),\ \textit{where}\ \widetilde{T_k} \in \widetilde{T}\ \textit{and}\ \widetilde{T_k}(x) = 1 &\iff \mu_{X/\widetilde{T}}(x,y) = \widetilde{T_k}(y),\ \textit{where}\ x \in A_k \\
&\iff \mu_{X/\widetilde{T}}(x,y) = \begin{cases} \alpha_k & \textit{if}\ s < k\ , \\ 1 & \textit{if}\ s = k\ , \\ \alpha_s & \textit{if}\ s > k\ . \end{cases} \\
&\qquad \textit{where}\ y \in A_s\ \textit{and}\ x \in A_k \\
&\iff \mu_{X/\widetilde{T}}(x,y) = \begin{cases} 1 & \textit{if}\ s = k\ , \\ \min\{\alpha_k, \alpha_s\} & \textit{if}\ s \neq k\ . \end{cases} \\
&\qquad \textit{where}\ y \in A_s\ \textit{and}\ x \in A_k
\end{aligned}$$

So we can guess that when we start with any function $f : X \longrightarrow [0,1]$, the following fuzzy relation, $\widetilde{R}$, becomes a fuzzy equivalence relation.

$$\mu_{\widetilde{R}}(x,y) = \begin{cases} 1 & \textit{if}\ f(x) = f(y)\ , \\ \min\{f(x), f(y)\} & \textit{if}\ f(x) \neq f(y)\ . \end{cases}$$

Example 8.1 *Now we are ready to make an S-H fuzzy partition on the unit circle given by $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. This can be done by using the ordinary partition of S given by $\{A_i : i \in \mathbb{N}\}$, where $A_i = \left\{(x,y) \in \mathbb{R}^2 : 1/(i+1) < \sqrt{x^2+y^2} \leq 1/i\right\}$, and also using the sequence given by $\{\alpha_i\}_{i=0}^{\infty}$, where $\alpha_i = 1/2^{i+1}$. Note that this is a decreasing sequence whose terms belong to the interval $[0, 0.5)$. So by the above theorem, $T := \{T_k : k \in \mathbb{N}\}$ given by*

$$T_i(x) = \begin{cases} 1/2^{i+1} & \textit{if}\ i > s\ , \\ 1 & \textit{if}\ i = s\ , \\ 1/2^{s+1} & \textit{if}\ i < s\ . \end{cases} = \begin{cases} 1 & \textit{if}\ i = s\ , \\ \min\{1/2^{i+1}, 1/2^{s+1}\} & \textit{if}\ k \neq s\ . \end{cases}$$

where $s \in \mathbb{N}$ such that $x \in A_s$, forms an S-H fuzzy partition.

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