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On LA-Rings of Finitely Nonzero Functions

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Abstract. Left almost semigroups (LA-semigroups) or Abel-Grassmann's groupoids (AG-groupoids) have been studied by several authors and this motivated to extend these concepts to Left Almost ring (LA-ring), which caries attraction due to its structural formation. In this paper we generalize the structure of commutative semigroup ring (ring of semigroup S over ring R represented as $R[X;S]$) to a non-associative LA-ring of commutative semigroup S over LA-ring R represented as $R[X^s; s\in S]$, consisting of finitely nonzero functions. Nevertheless it also possesses associative ring structures. Furthermore we also discuss the LA-ring homomorphisms.

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1. Introduction

Assume that $(R,+,\cdot)$ is an associative ring and $(S,*)$ is a semigroup. Let J be the set of all finitely nonzero functions f from S into R. Suppose J is a ring with respect to binary operations addition and multiplication defined as

$$\begin{aligned}(f+g)(s) &= f(s)+g(s)\\ (fg)(s) &= \sum_{t*u=s} f(t)g(u),\end{aligned}$$

where the symbol $\sum\limits_{t*u=s}$ indicates that the sum is taken over all pairs (t,u) of elements of S such that $t*u=s$ and it is understood that in the situation where s is not expressible in the form $t*u$ for any $t,u\in S$, then $(fg)(s)=0$. J is known as *semigroup ring* of S over R. If S is a monoid, then J is called monoid ring. This ring J is represented as $R[S]$ whenever S is a multiplicative semigroup and elements of J are written either as $\sum_{s\in S} f(s)s$ or as $\sum_{i=1}^{n} f(s_i)s_i$. The representation of J will be $R[X;S]$ whenever S is an additive semigroup. As there is an isomorphism between additive semigroup S and multiplicative semigroup $\{X^s : s \in S\}$, so a nonzero element f of $R[X;S]$ is uniquely represented in the canonical form $\sum_{i=1}^{n} f(s_i)X^{s_i} = \sum_{i=1}^{n} f_i X^{s_i}$, where $f_i \neq 0$ and $s_i \neq s_j$ for $i \neq j$.

The concepts of degree and order are not generally defined in semigroup rings. But if we consider S to be a totally ordered semigroup, we can define the degree and order of an element of semigroup ring $R[X;S]$ in the following manner; if $f = \sum_{i=1}^{n} f_i X^{s_i}$ is the canonical form of the nonzero element $f \in R[X;S]$, where $s_1 < s_2 < \ldots < s_n$, then s_n is called the degree of f and we write $\deg(f) = s_n$ and similarly the order of f is written as $\operatorname{ord}(f) = s_1$. Now, if R is an integral domain, then for $f, g \in R[X;S]$, we have

$$\begin{aligned} \deg(fg) &= \deg(f) + \deg(g) \\ \operatorname{ord}(fg) &= \operatorname{ord}(f) + \operatorname{ord}(g). \end{aligned}$$

If S is Z_0 and R is an associative ring, the semigroup ring J is simply the polynomial ring $R[X]$. So a polynomial ring is a particular case of the semigroup ring.

Left almost semigroups (LA- semigroups) or Abel-Grassmann's groupoids (AG-groupoids) have been discussed in [2], [4], [5], [7] and [8].

In [4], M. Kazim and M. Naseerudin introduced the notion of left almost semigroup (LA-semigroup) which is a generalization of commutative semigroup. This structure is also known as Abel-Grassmann's groupoid (AG-groupoid) (see [2]). A groupoid $(S, *)$ is called a *left almost-semigroup (LA-Semigroup)* or *Abel-Grassmann's groupoid (AG-groupoid)*, if it satisfies the *left invertive law;* $(a * b) * c = (c * b) * a$ for all $a, b, c \in S$. For example, let $(\mathbb{Z}, +)$ denote the group of integers which is an LA-semigroup under the binary operation "$*$", defined as; $a * b = b - a$, where "$-$" denotes the ordinary subtraction defined in $\mathbb{Z}$. By [4, Proposition 2.1], every LA-semigroup $(S, *)$ is medial that is

$$(a * b) * (c * d) = (a * c) * (b * d) \text{ for all } a, b, c, d \in S.$$

Q. Mushtaq and M. S. Kamran extended the notion of left almost semigroup (LA-semigroup) to left almost group (LA-group) (see [6]). A groupoid $(G, *)$ is called a left almost group (LA-group), if (i) there exists left identity $e \in G$ such that $e * a = a$ for all $a \in G$, (ii) for $a \in G$ there exists $b \in G$ such that $b * a = e$ and (iii) left invertive law holds.

It is important to note that if $a^{'}$ is left inverse of a in LA-group with left identity e then $a^{'}$ also becomes right inverse. Indeed,

$$a * a^{'} = (e * a) * a^{'} = (a^{'} * a) * e = e * e = e.$$

It can also be shown that every commutative semigroup implies an LA-semigroup. Indeed,

$$(a * b) * c = c * (a * b) = c * (b * a) = (c * b) * a \text{ for all } a, b, c \in S.$$

Recently in [9], S. M. Yusuf extended these notions to a non-associative structure with respect to both binary operations "+" and "·" namely left almost ring (LA-ring). By a left almost ring, we mean a non empty set R such that $(R, +)$ is an LA-group and $(R, \cdot)$ is an LA-semigroup and both left and right distributive laws hold. For example, from a commutative ring $(R, +, \cdot)$, we can always obtain an LA-ring $(R, \oplus, \cdot)$ by defining, for $a, b, c \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. We can not assume the addition to be commutative in an LA-ring. An LA-ring $(R, +, \cdot)$ is said to be LA-integral domain if $ab = 0$, $a, b \in R$, then $a = 0$ or $b = 0$. Let $(R, +, \cdot)$ be an LA-ring and S be a non-empty subset of R and S is itself an LA-ring under the binary operations induced by R, then S is called an LA-subring of $(R, +, \cdot)$. If S is an LA-subring of an LA-ring $(R, +, \cdot)$, then S is called a left ideal of R if $RS \subseteq S$. Right and two-sided ideals are defined in the usual manner.

In this study, we generalize the structure of a commutative semigroup ring (ring of functions from a commutative semigroup S to ring R represented as $R[X; S]$) to an LA-ring of commutative semigroup S over LA-ring R represented as $R[X^s; s \in S]$, which is a non-associative structure, consisting of finitely nonzero functions from a commutative semigroup S into LA-ring R. Generally, the concepts of degree and order are not defined in semigroup rings unless we consider S, a totally ordered semigroup with 0 adjoined. Analogous to commutative semigroup rings $R[X; S]$, we may define degree and order of an element of LA-ring $R[X^s; s \in S]$.

2. The Construction

Let $(R, +, \cdot)$ be an LA-ring with left identity and S be a commutative semigroup under binary operation "$*$". Let $T = \{f \mid f : S \to R, \text{ where } f \text{ are finitely nonzero}\}$.

Define the binary operation "+" in T as $(f + g)(s) = f(s) + g(s)$.

$(T, +)$ is an LA-group. Indeed, let $f, g \in T$. Now as $f(s), g(s) \in R$ for all $s \in S$, so, $(f + g)(s) = f(s) + g(s) \in R$ and hence $f + g \in T$.

Let $f,g,h\in T$. As $f(s),g(s),h(s)\in R$, so by left invertive law in $(R,+)$, we have

$$\begin{aligned}((f+g)+h)(s) &= (f+g)(s)+h(s)\\ &= (f(s)+g(s))+h(s)\\ &= (h(s)+g(s))+f(s)\\ &= (h+g)\,(s)+f(s)\\ &= ((h+g)+f)(s).\\ \text{Hence } (f+g)+h &= (h+g)+f.\end{aligned}$$

Thus left invertive law holds in T.
Define the map $o:S\rightarrow R$ such that $o(s)=0$ for all $s\in S$,

$$\begin{aligned}(o+f)(s) &= o(s)+f(s)=0+f(s)=f(s)\\ o+f &= f.\end{aligned}$$

Thus o is left additive identity in T.
For every $f\in T$ there exists a function $-f:S\rightarrow R$ defined by $(-f)(s)=-f(s)$ for all $s\in S$ and

$$\begin{aligned}((-f)+f)(s) &= (-f(s))+f(s)\\ &= -f(s)+f(s)\\ &= 0=o(s)\\ (-f)+f &= o.\end{aligned}$$

So the left inverses exist in $(T,+)$. Hence $(T,+)$ is an LA-group. We can say $f+(-f)=0$ as $-f(s)$ is also the right inverse of $f(s)$ in R by [6]. Now we define binary operation "$\odot$" in T as follows

$$f\odot g\,(s)=\sum_{t*u=s}f(t)\cdot g(u).$$

We claim that $(T,\odot)$ is an LA-semigroup. As for $f(t)$ and $g(u)\in R$, where $t,u\in(S,*)$ and $(R,\cdot)$ is LA-ring, $f\odot g(s)\in R$. Since f,g are finitely nonzero on S, therefore $f\odot g\in T$. For $f,g,h\in T$ and $s\in S$, consider

$$\begin{aligned}
[(f\odot g)\odot h](s) &= \sum_{t*u=s}(f\odot g)(t)\cdot h(u)\\
&= \sum_{t*u=s}\{\sum_{t=p*q}((f(p)\cdot g(q))\}\cdot h(u)\\
&= \sum_{(p*q)*u=s}(f(p)\cdot g(q))\cdot h(u)\\
&= \sum_{(u*q)*p=s}(h(u)\cdot g(q))\cdot f(p).
\end{aligned}$$

As every commutative semigroup implies an LA-semigroup, so $(p*q)*u=(u*q)*p$ for all $p,q,u\in(S,*)$.

Hence

$$\begin{aligned}
[(f\odot g)\odot h](s) &= \sum_{(p*q)*u=s}(f(p)\cdot g(q))\cdot h(u)\\
&= \sum_{(u*q)*p=s}(h(u)\cdot g(q))\cdot f(p)\\
&= \sum_{r^{'}*p=s}\{\sum_{r^{'}=u*q}(h(u)\cdot g(q))\}\cdot f(p)\\
&= \sum_{r^{'}*p=s}(h\odot g)(r^{'})\cdot f(p)\\
&= \left[(h\odot g)\odot f\right](s).
\end{aligned}$$

Thus $(T,\odot)$ is an LA-semigroup. Now we verify that the binary operation "$\odot$" is distributive over addition. Indeed as $f(t), g(u)$ and $h(u)\in R$ and multiplication is distributive over addition in R, so

$$\begin{aligned}
\left[f\odot(g+h)\right](s) &= \sum_{t*u=s}f(t)\cdot(g+h)(u)\\
&= \sum_{t*u=s}f(t)\cdot(g(u)+h(u))\\
&= \sum_{t*u=s}(f(t)\cdot g(u)+f(t)\cdot h(u))\\
&= \sum_{t*u=s}f(t)\cdot g(u)+\sum_{t*u=s}f(t)\cdot h(u)\\
&= (f\odot g)(s)+(f\odot h)(s)\\
&= \left[f\odot g+f\odot h\right](s).\\
\text{Hence } f\odot(g+h) &= f\odot g+f\odot h.
\end{aligned}$$

Similarly

$$\begin{aligned}
\left[(g+h)\odot f\right](s) &= \sum_{t*u=s}(g+h)(t)\cdot f(u) \\
&= \sum_{t*u=s}(g(t)+h(t))\cdot f(u) \\
&= \sum_{t*u=s}(g(t)\cdot f(u)+h(t)\cdot f(u)) \\
&= \sum_{t*u=s}g(t)\cdot f(u)+\sum_{t*u=s}h(t)\cdot f(u) \\
&= (g\odot f)(s)+(h\odot f)(s) \\
&= [g\odot f+h\odot f](s). \\
\text{Hence } (g+h)\odot f &= g\odot f+h\odot f.
\end{aligned}$$

Thus $(T,+,\odot)$ is an LA- ring of commutative semigroup $(S,*)$ over LA-ring $(R,+,\cdot)$.

Remark 1. If we take $S=\mathbb{Z}_0$ then polynomial LA-ring becomes a particular case of LA-ring $(T,+,\odot)$.

2.1. **Representation of elements of T.** To represent the elements of ring $(T,+,\odot)$, we first define LA- modules over an LA-ring R.

Definition 1. Let $(R,+,.)$ be an LA-ring with left identity. An LA-group $(M,+)$ is said to be LA-module over R if $R\times M\to M$ defined as $(a,m)\mapsto am\in M$, where $a\in R$, $m\in M$ satisfies

(1) $(a+b)m=am+bm$;
(2) $a(m+n)=am+an$;
(3) $a(bm)=b(am)$;
(4) $1.m=m$,
for all $a,b\in R$, $m,n\in M$.

For instance, let $(R,+,\cdot)$ be an LA-ring with left identity and S be a commutative semigroup. It is important to note that every commutative semigroup is an LA-semigroup. Now it is easy to verify that $R[S]=\{\sum\limits_{j=1}^{n}a_js_j : a_j\in R, s_j\in S\}$ is an additive LA-group. We claim that $R[S]$ is an LA-module over R. Indeed, let $R\times R[S]\mapsto R[S]$ be defined as $(a,\sum\limits_{j=1}^{n}a_js_j)\mapsto\sum\limits_{j=1}^{n}(aa_j)s_j$ which is obviously well-defined.

The first two and fourth properties are easy to verify. We verify the third property.

$$\begin{aligned}
\text{Consider } a(b\sum_{j=1}^{n} a_j s_j) &= a(\sum_{j=1}^{n} (ba_j)\, s_j) \\
&= (\sum_{j=1}^{n} (a\,(ba_j))\, s_j)
\end{aligned}$$

As R is an LA-ring with left identity, so by [7, Lemma 4], $a(bc) = b(ac)$ holds for all $a, b, c \in R$.

$$\begin{aligned}
\text{Hence } a(b\sum_{j=1}^{n} a_j s_j) &= (\sum_{j=1}^{n} (b\,(aa_j))\, s_j) \\
&= b(\sum_{j=1}^{n} (aa_j)\, s_j) \\
&= b(a\sum_{j=1}^{n} a_j s_j) \\
\text{Thus } a(b\sum_{j=1}^{n} a_j s_j) &= b(a\sum_{j=1}^{n} a_j s_j).
\end{aligned}$$

Remark 2. If (S,$\cdot$) is a commutative semigroup, then $T = R[S]$ and elements of LA-ring T are written either in the form of $\sum_{s\in S} f(s)s$ or $\sum_{i=1}^{n} f(s_i)s_i$. Thus (1) S is a free basis for $R[S]$ as an LA-module over LA-ring R and (2) multiplication in $R[S]$ is determined by using distributivity and by setting $(r_1s_1)(r_2s_2) = (r_1r_2)(s_1s_2)$ where $r_1, r_2 \in R$ and $s_1, s_2 \in S$. Indeed, consider

$$f(s_1)s_1 + f(s_2)s_2 + ... + f(s_n)s_n = 0.$$

Since $s_1, s_2, ..., s_n \in S$ and $s_i \neq 0$ for all i=$1, 2, ..., n$, so, $f(s_i) = 0$ for all $i = 1, 2, ..., n$. Thus $s_1, s_2, ..., s_n$ are linearly independent. Now let $f(s_1), f(s_1), ..., f(s_n) \in R$ and $s_1, s_2, ..., s_n \in S$. Then, $f(s_1)s_1 + f(s_2)s_2 + ... + f(s_n)s_n$ is a linear combination of elements of S whose coefficients are from the LA-ring R. Thus S is a free basis for ring T as an LA-module over an LA-ring R. Now let $f = \sum_{i=1}^{n} f(s_i)s_i$ and

$g=\sum_{i=1}^{m}g(t_i)t_i$.

$$\begin{aligned}
For\ n &= 1, m=1,\ f\cdot g=(f(s_1)s_1)\cdot(g(t_1)t_1)\\
&= f(s_1)g(t_1)\cdot s_1t_1\\
For\ n &= 2, m=2,\ (f(s_1)s_1+f(s_2)s_2)\cdot(g(t_1)t_1+g(t_2)t_2)\\
&= f(s_1)s_1\cdot(g(t_1)t_1+g(t_2)t_2)+f(s_2)s_2\cdot(g(t_1)t_1+g(t_2)t_2)\\
&= f(s_1)s_1\cdot g(t_1)t_1+f(s_1)s_1\cdot g(t_2)t_2+f(s_2)s_2\cdot g(t_1)t_1+f(s_2)s_2\cdot g(t_2)t_2\\
&= f(s_1)g(t_1)\cdot s_1t_1+f(s_1)g(t_2)\cdot s_1t_2+f(s_2)g(t_1)\cdot s_2t_1+f(s_2)g(t_2)\cdot s_2t_2\\
&= \sum_{i+j=2}^{4}f(s_i)g(t_j)s_it_j.\text{ Thus in general, } f\cdot g=\sum_{i+j=2}^{m+n}f(s_i)g(t_j)s_it_j.
\end{aligned}$$

Remark 3. If $(S,+)$ is a commutative semigroup, then the elements of T are written either in the form $\sum_{s\in S}f(s)X^s$ or $\sum_{i=1}^{n}f(s_i)X^{s_i}$.

From next lemma, it is obvious that the introduction of symbol X and notation X^s has the effect of transforming $(S,+)$ into $(\{X^s \mid s\in S\},\cdot)$ by means of isomorphism.

Lemma 1. *For a semigroup $(S,+)$, there exists a semigroup $(\{X^s : s\in S\},\cdot)$ which is isomorphic to S, where "$\cdot$" is usual multiplication.*

Thus by the effect of this isomorphism, the representation of an element f of T gets the form $f=\sum_{i=1}^{n}f(s_i)X^{s_i}$ or $\sum_{i=1}^{n}f_iX^{s_i}$, where $f_i=f(s_i)$. We shall represent T by $R[X^s; s\in S]$.

2.2. **Degree and order of elements of LA-ring $R[X^s; s\in S]$.** The concepts of degree and order are not generally defined in semigroup rings unless we have to consider S, a totally ordered semigroup with 0 adjoined (that is ordered monoid). The structure of LA-ring $R[X^s; s\in S]$ is also not convenient for defining degree and order of an element unless $(S,*)$ is totally ordered. Here we define support of $f=\sum_{i=1}^{n}f_iX^{s_i}$ abbreviated as $\mathrm{Supp}(f)=\{s_i : f_i\neq 0\}$. The order and degree of f is defined as $\mathrm{ord}(f)=\min(\mathrm{supp}(f))$ and $\deg(f)=\max(\mathrm{supp}(f))$.

Lemma 2. *(1) If R is an LA-ring with left identity, then for $f,g\in R[X^s; s\in S]$, $deg(f.g)\leq deg(f)+deg(g)$.*

(2) If R is an LA-integral domain, then $deg(f.g)=deg(f)+deg(g)$.

Proof. (1) Let $f=(f_0+f_1X+...,f_nX^n)$ and $g=(g_0+g_1X+...,g_mX^m)$ where $f_n,g_m\neq 0$. So

$$f\cdot g = (f_0g_0,(f_1g_0+f_0g_1)X+...+f_ng_mX^{n+m}.$$

Now if $f_n\cdot g_m \neq 0$, then, $\deg(f\cdot g)=n+m=\deg(f)+\deg(g)$ and if $f_n\cdot g_m=0$, then $\deg(f\cdot g) < \deg(f)+\deg(g)$. Thus $\deg(f.g)\leq\deg(f)+\deg(g)$.

(2) As R is an LA-integral domain, so for $f_n \neq 0, g_m \neq 0$, the product $f_n \cdot g_m \neq 0$. Thus clearly deg$(f.g)$ =deg(f) +deg(g) . ∎

3. Further Developments

In [9], a mapping φ of an LA-ring $(R, +, \cdot)$ into an LA-ring $(R', +, \cdot)$ is called a homomorphism if $\varphi(a+b) = \varphi(a) + \varphi(b)$ and $\varphi(ab) = \varphi(a) \cdot \varphi(b)$ and for the two sided ideal I of R, the mapping $\nu : R \to R/I$ defined as $\nu(a) = a + I$ is called the natural epimorphism of LA-ring R onto R/I.

Let θ be an epimorphism of an LA-ring R to an LA-ring R', then $R/Ker\theta \simeq R'$.

The following is a generalization of [1, Theorem 7.1].

Theorem 1. *Let R be an LA-ring and L, M be commutative semigroups. Then*

$$R\left[X^{(l,m)}; (l,m) \in L \oplus M\right] \simeq (R\left[X^{l}; l \in L\right])([X^{m}; m \in M]),$$

where $L \oplus M$ the is external direct sum of commutative semigroups L and M.

Proof. Here we regard the elements of LA- ring R, as finitely non-zero functions from commutative semigroup to LA-ring R.

Assume that $(R\left[X^{l}; l \in L\right])([X^{m}; m \in M]) = A$. Define $\phi : A \to R\left[X^{(l,m)}; (l,m) \in L \oplus M\right]$ by $[\phi(f)](l,m) = [f(m)](l)$ where $f(m) \in R[X^{l}; l \in L]$, $f \in A$ and $f : M \to R[X^{l}; l \in L]$. Clearly ϕ is surjective because if $h \in R\left[X^{(l,m)}; (l,m) \in L \oplus M\right]$, then the element $f \in A$ defined by

$$\begin{aligned} [f(m)](l) &= h(l,m) \text{ is such that} \\ [\phi(f)](l,m) &= [f(m)](l) = h(l,m). \end{aligned}$$

$$\begin{aligned} \text{Now Suppose } f &\neq g. \\ \text{Then } f(m) &\neq g(m) \text{ for some } m \in M. \\ [f(m)](l) &\neq [g(m)](l) \text{ for some } l \in L, m \in M \\ \phi(f)(l,m) &\neq \phi(g)(l,m) \\ \phi(f) &\neq \phi(g). \end{aligned}$$

Thus ϕ is one-one. It is an LA-ring homomorphism. Indeed, let $f, g \in A$ and $(l,m) \in L \oplus M$.

$$\begin{aligned} [\phi(f+g)](l,m) &= [(f+g)(m)](l) \\ &= [f(m) + g(m)](l) \\ &= [f(m)](l) + [g(m)](l) \\ &= [\phi(f)](l,m) + [\phi(g)](l,m) \\ &= [\phi(f) + \phi(g)](l,m) \\ \text{Hence } \phi(f+g) &= \phi(f) + \phi(g). \end{aligned}$$

Now consider

$$\begin{aligned}
[\phi(f\odot g)](l,m) &= [(f\odot g)(m)](l) \\
&= [\sum_{a*b=m}(f(a)\cdot g(b))](l) \\
&= \sum_{a*b=m}(\sum_{c*d=l}(f(a))(c))\cdot((g(b))(d)) \\
&= \sum_{(c*d,a*b)=(l,m)}(f(a))(c))\cdot((g(b))(d) \\
&= \sum_{(c*a)+(d*b)=(l,m)}\phi(f)(c,a)\cdot\phi(g)(d,b) \\
&= [\phi(f)\odot\phi(g)](l,m) \text{ for all } (l,m)\in L\oplus M. \\
\text{Hence } \phi(f\odot g) &= \phi(f)\odot\phi(g).
\end{aligned}$$

Thus ϕ is an isomorphism. ∎

Remark 4. *From above result, for the finite set $\{Y_i\}_{i=1}^n$ of indeterminates, isomorphism of $R[Y_1, Y_2, ..., Y_n]$ and $R[X;\mathbb{Z}_0^n]$ follows by induction.*

The following theorem is an extended form of theorem 1.

Theorem 2. *The polynomial LA-ring $R[\{Y_\lambda\}_{\lambda\in\Lambda}]$, where R is an LA-ring and $\{Y_\lambda\}_{\lambda\in\Lambda}$ is a family of commuting indeterminates and $F=\sum_{\lambda\in\Lambda}^{w} Z_\lambda$ such that $Z_\lambda \simeq Z_0$. Then $R[\{Y_\lambda\}_{\lambda\in\Lambda}]$ is isomorphic to LA-ring $R[X;F]$ of free commutative semigroup $(F,+)$ over R.*

Proof. As $F=\sum_{\lambda\in\Lambda}^{w} Z_\lambda$ such that $Z_\lambda\simeq Z_0$ for each λ and $\{e_\lambda\}_{\lambda\in\Lambda}$ be the standard free basis for F that is the λ-th coordinate of e_λ is 1 and all others coordinates are 0. Each element of F is uniquely expressible in the form $a=\sum k_\lambda e_\lambda$, for some $k_\lambda\geq 0\,(k_\lambda\in Z_\lambda)$. For each $r_aX^a\in R[X;F]$, we have
$\sum r_a\Pi_{\lambda\in\Lambda}Y_\lambda^{k_\lambda}\in R[\{Y_\lambda\}_{\lambda\in\Lambda}]$. We define

$$\phi: R[X;F]\rightarrow R\left[[Y_\lambda]_{\lambda\in\Lambda}\right] \text{ by } \phi(\sum_{a\in F} r_aX^a)=\sum_{a\in F} r_a\Pi_{\lambda\in\Lambda}Y_\lambda^{k_\lambda}.$$

$$\begin{aligned}
\text{Suppose } \sum_{a\in F} r_a X^a &= \sum_{b\in F} r_b X^b. \\
\text{Here, } a &= \sum k_\lambda e_\lambda \text{ and } b = \sum k'_\lambda e_\lambda, \text{ where } k_\lambda, k'_\lambda \geq 0. \\
\sum_{a\in F} r_a X^{\Sigma k_\lambda e_\lambda} &= \sum_{b\in F} r_b X^{\Sigma k'_\lambda e_\lambda} \\
\sum_{a\in F} r_a X^{k_1e_1+k_2e_2+\ldots+k_\lambda e_\lambda+\ldots} &= \sum_{b\in F} r_b X^{k'_1e_1+k'_2e_2+\ldots+k'_\lambda e_\lambda+\ldots} \\
\sum_{a\in F} r_a X^{k_1e_1} X^{k_2e_2} \ldots X^{k_\lambda e_\lambda} \ldots &= \sum_{b\in F} r_b X^{k'_1e_1} X^{k'_2e_2} \ldots X^{k'_\lambda e_\lambda} \ldots \\
\sum_{a\in F} r_a X^{(k_1,0,0,\ldots)} \ldots X^{(\ldots,0,0,0,k_\lambda\ldots)} \ldots &= \sum_{b\in F} r_b X^{(k'_1,0,0,\ldots)} \ldots X^{(0,\ldots,0,k'_\lambda,0\ldots)} \ldots \\
\sum_{a\in F} r_a Y_1^{k_1} . Y_2^{k_2} \ldots Y_\lambda^{k_\lambda} \ldots &= \sum_{b\in F} r_b Y_1^{k'_1} . Y_2^{k'_2} \ldots Y_\lambda^{k'_\lambda} \ldots \\
\sum_{a\in F} r_a \Pi_{\lambda\in\Lambda} Y_\lambda^{k_\lambda} &= \sum_{b\in F} r_b \Pi_{\lambda\in\Lambda} Y_\lambda^{k'_\lambda} \\
\phi(\sum_{a\in F} r_a X^a) &= \phi(\sum_{b\in F} r_b X^b).
\end{aligned}$$

Thus ϕ is well-defined. Now it is straight forward to prove that ϕ is an isomorphism. ∎

The following is a generalized form of [1, Theorem 8.1].

Theorem 3. *Let R be an LA-ring with left identity e and let $(S, *)$ be a commutative semigroup. Let $R[X^s; s \in S]$ be an LA-ring of S over R. Then $R[X^s; s \in S]$ is an LA-integral domain if and only if R is an LA-integral domain and S is an torsion free and cancellative.*

Proof. Assume that R is an LA-integral domain. Now S is torsion free and cancellative if and only if S admits a total order $\leq$ compatible with its operation (see [1, corollary 3.4]).

$$\begin{aligned}
\text{Let } f, g &\in R[X^s; s \in S] \setminus \{0\} \text{ such that} \\
f &= \sum_{i=1}^{m} f_i X^{s_i}, g = \sum_{i=1}^{n} g_i X^{t_i}, \\
\text{where } s_1 &\leq s_2 \leq \ldots \leq s_m \text{ and } t_1 \leq t_2 \leq \ldots \leq t_n.
\end{aligned}$$

If $f_1 \neq 0, g_1 \neq 0$, then $s_1 + t_1 \in$supp$(f \odot g)$ and $f_1 g_1 X^{s_1+t_1}$ is the corresponding term in $f \odot g$. In particular, $f \cdot g \neq 0$ hence $R[X^s; s \in S]$ is an LA-integral domain. Conversely, assume that $R[X^s; s \in S]$ is an LA-integral domain. On the contrary, suppose that R is not an LA- integral domain then for $a, b \in R\backslash\{0\}$, we have $a.b = 0$. If $s \in S$, then $aX^s \odot bX^s = 0$, where $aX^s \neq 0, bX^s \neq 0$. This implies that $R[X^s; s \in S]$ is not an LA-integral domain. Similarly, if S is not cancellative and $s, t, u \in S$ are such that $s + t = s + u$ but $t \neq u$, then for $r \in R\backslash\{0\}$, we have $rX^s \odot (rX^t - rX^u) = 0$, where rX^s and $rX^t - rX^u$ are nonzero. Hence $R[X^s; s \in S]$ is not an LA-integral domain. Finally, assume that R is an LA-integral domain and that S is cancellative but not torsion free. Let $s, t \in S$ be such that $s \neq t$ while $ns = nt$ for some $n \in \mathbb{Z}^+$ and choose $k \in \mathbb{Z}^+$ minimal so that $ks = kt$.

$$\begin{aligned} \text{If } 0 &\neq r \in R, \text{ then } 0 = (r^2X^{ks} - r^2X^{kt}) \\ &= (rX^s - rX^t) \odot (\sum_{i=0}^{k-1} rX^{(k-i-1)s+it}). \end{aligned}$$

Since S is cancellative the choice of k implies

$$\begin{aligned} (k - i_1 - 1)s + i_1t &\neq (k - i_2 - 1)s + i_2t \text{ for } 0 \leq i_1 < i_2 \leq k - 1. \\ \text{Thus } \sum_{i=0}^{k-1} rX^{(k-i-1)s+it} &\neq 0 \end{aligned}$$

Hence again $R[X^s; s \in S]$ is not an LA-integral domain. ∎

The next result generalizes [1, Theorem 7.2] and contains some basic informations concerning homomorphisms of LA-rings of commutative semigroups over LA-rings.

Theorem 4. *Let $\mu : R \longrightarrow R_0$ be an LA-ring homomorphism. Let $A =$kerμ and $\phi : S \longrightarrow S_0$ be a semigroup homomorphism, where S, S_0 are commutative semigroups with 0 adjoined to them. Then the following statements holds;*

(1) *$\mu^* : R[X^s; s \in S] \longrightarrow R_0[X^s; s \in S]$ is defined as $\mu^*(\sum_{i=1}^n r_iX^{s_i}) = \sum_{i=1}^n \mu(r_i) X^{s_i}$, is LA-ring homomorphism such that ker$\mu^* = A[X^s; s \in S] =ker\mu[X^s; s \in S]$. μ^* is surjective if μ is surjective.*

(2) *$\phi^* : R[X^s; s \in S] \rightarrow R[X^s; s \in S_0]$ defined as $\phi^*(\sum_{i=1}^n r_iX^{s_i}) = \sum_{i=1}^n r_iX^{\phi(s_i)}$, is surjective and ker$\phi^* = I$. The ideal of $R[X^s; s \in S]$ generated by $\{rX^a - rX^b : \phi(a) = \phi(b), r \in R\}$. ϕ^* is surjective if ϕ is surjective.*

(3) *$\tau : R[X^s; s \in S] \rightarrow R_0[X^s; s \in S_0]$ defined as $\tau(\sum_{i=1}^n r_iX^{s_i}) = \sum_{i=1}^n \mu(r_i) X^{\phi(s_i)}$, is an LA-ring homomorphism such that ker$\tau =$ kerμ^*+ker$\phi^* = A[X^s; s \in S] + I$. Then τ is surjective if μ and ϕ are surjective.*

Proof. Same as in [1, Theorem 7.2]. ∎

Corollary 1. *Assume that A is an ideal of the LA-ring R and that $\sim$ is a congruence on commutative semigroup $S = S \cup \{0\}$. Let $I = \{rX^a - rX^b : \phi(a) = \phi(b), r \in R\}$, $\phi : S \rightarrow S/\sim$ is a canonical epimorphism. Then $R[X^s; s \in S]/A[X^s; s \in S] \simeq \frac{R}{A}[X^s; s \in S]$ and $R[X^s; s \in S]/I \simeq R[X^s; s \in S/\sim]$, where the ideal I is called the kernel ideal of congruence.*

Proof. As $\phi : S \to S/\sim$ is defined by $\phi(s) = [s]$. So we define $\phi^* : R[X^s; s \in S] \to R[X^s; s \in S/\sim]$ as $\phi^*(\sum_{i=1}^n r_i X^{s_i}) = \sum_{i=1}^n r_i X^{\phi(s_i)} = \sum_{i=1}^n r_i X^{[s_i]}$ which is an LA-ring epimorphism. Therefore

$$R[X^s; s \in S]/\ker\phi^* \simeq R[X^s; s \in S/\sim].$$

Similarly we may define a map

$$\mu^* : R[X^s; s \in S] \to \frac{R}{A}[X^s; s \in S] \text{ by } \mu^*(\sum_{i=1}^n r_i X^{s_i}) = \sum_{i=1}^n \mu(r_i) X^{s_i},$$

where $\mu : R \to \frac{R}{A}$ is a surjective LA-ring homomorphism defined as $\mu(r) = r + A$. μ^* is also LA- ring epimorphism. Hence $R[X^s; s \in S]/\ker\mu^* \simeq \frac{R}{A}[X^s; s \in S]$.

$$\begin{aligned} \text{It can be shown that } \ker\mu^* &= A[X^s; s \in S]. \\ \text{Therefore, } R[X^s; s \in S]/A[X^s; s \in S] &\simeq \frac{R}{A}[X^s; s \in S]. \end{aligned}$$

∎

Definition 2. An ideal P of an LA-ring R is called prime if and only if $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$, where A and B are ideals in R.

Theorem 5. *Let R be an LA-ring with left identity 1. Then P is a prime ideal in R if and only if R/P is an LA-integral domain having the left identity $1 + P$.*

Proof. Same as [3, Theorem 2.16]. ∎

The following is a generalized form of [1, Corollary 8.2].

Corollary 2. *Let A be a proper ideal of LA-ring R, then $A[X^s; s \in S]$ is prime ideal in $R[X^s; s \in S]$ if and only if A is prime ideal in R and S is cancellative and torsion free semigroup.*

Proof. Follows from theorem 3 and corollary 1. ∎

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