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On Pre-bg-Closed Functions and Associated Properties

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Abstract. The concept of b-open sets was introduced by Andrijevic. The primary purpose of this paper is to introduce and study pre-bg-closed functions by using b-open sets.

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1. Introduction and Preliminaries

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the variously modified forms of continuity, seperation axioms etc. by utiliaing generalized open sets. One of the most well known notions and also an inspiration source is the notion of b-open [1] sets introduced by Andrijevic in 1996.

This class is a subclass of the class of semi preopen sets [2], that is a subset of a topological space which is contained in the closure of the interior of its closure. Also, the class of b-open sets is a superset of the class of semi-open sets [9], that is a set which is contained in the closure of its interior, and the class of locally dense sets [4] or preopen sets [10], that is a set which is contained in the interior of its closure. Andrijevic studied several fundamental and interesting properties of b-open sets. Among others, he showed that a rare b-open set is preopen [1]. Recall that a rare set [3] is a set with no interior points. It is also well-known that for a topological space X, every preopen set is semi-open if and only if the interior of a dense subset is dense.

Throughout this paper, X and Y refer always to topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A and the interior of A in X, respectively. A subset A of X is said to be b-open [1] ($= \gamma$-open [7]) (resp. regular open [12]) $A \subset \mathrm{Cl}(\mathrm{Int}(A)) \cup \mathrm{Int}(\mathrm{Cl}(A))$ ($A = \mathrm{Int}(\mathrm{Cl}(A))$). The complement of b-open (resp. regular open) set is called b-closed ($= \gamma$-closed) (resp. regular closed). The intersection of all b-closed sets of X containing A is called the b-closure [1] ($= \gamma$-closure) of A and is denoted by $b\,\mathrm{Cl}(A)$ ($= \gamma$-$\mathrm{Cl}(A)$), it is evident that a set A is b-closed if and only if $b\,\mathrm{Cl}(A) = A$. The b-interior [1] of A, $b\,\mathrm{Int}\,(A)$, is the union of all b-open sets contained in A. A set A is said to be b-regular [11] if it is b-open and b-closed. The family of all b-open (resp. b-closed, b-regular, regular open, regular closed) sets of X is denoted by $BO(X)$ (resp. $BC(X), BR(X), RO(X), RC(X)$). A subset A of a space X is called a γ-generalized closed [8] (briefly γg-closed $= bg$-closed) set of X if γ-$\mathrm{Cl}(A) \subset U$ holds whenever $A \subset U$ and U is open in X, A will be called bg-open if $X\backslash A$ is bg-closed.

2. Pre-bg-closed functions

Definition 2.1. *A function $f : X \to Y$ is said to be pre-bg-closed (resp. regular bg-closed, almost bg-closed) if for each $F \in BC(X)$ (resp. $F \in BR(X), F \in RC(X)$), $f(F)$ is bg-closed in Y.*

Definition 2.2. *A function $f : X \to Y$ is said to be bg-closed if for each closed set F of X, $f(F)$ is bg-closed in Y.*

From the above definitions, we obtain the following diagram:

$$\begin{array}{ccc} \text{pre-}bg\text{-closed} & \to & \text{regular } bg\text{-closed} \\ \downarrow & & \downarrow \\ bg\text{-closed} & \to & \text{almost } bg\text{-closed} \end{array}$$

Remark 2.3. *None of all implications in the above diagram is reversible as the following three examples show.*

Example 2.4. *Let $X = \{a, b, c\}$ with topologies $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{a, b\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is bg-closed and thus almost bg-closed (observe that $\{a\}$ is b-closed in (X, σ)). However, f is not regular bg-closed and thus not pre-bg-closed (observe that $\{a, b\}$ is b-regular in (X, τ) but not bg-closed in (X, σ) as $\{a, b\} \subset \{a, b\}$ but $b\,\mathrm{Cl}(\{a, b\}) = X \not\subset \{a, b\}$).*

Example 2.5. *Let $X = \{a, b, c\}$ with topologies $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{a\}, \{a, b\}, X\}$. Define a function $f : (X, \tau) \to (X, \sigma)$ by $f(a) = c$, $f(b) = a$ and $f(c) = b$. Then f is almost bg-closed (obsrve that the regular closed subsets of (X, τ) are X and ϕ, each of which is clearly bg-closed in (X, σ). However, f is not bg-closed as $\{b, c\}$ is closed in (X, τ) but $f(\{b, c\}) = \{a, b\}$ is not bg-closed in (X, σ) (observe that $\{a, b\} \subset \{a, b\}$ but $b\,\mathrm{Cl}(\{a, b\}) = X \not\subset \{a, b\}$).*

Example 2.6. *Let $X = \{a, b, c\}$ with topologies $\tau = \{\phi, \{a, b\}, X\}$ and $\sigma = \{\phi, \{c\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is regular bg-closed (observe that each subset of X except $\{a, b\}$ and $\{c\}$ is b-regular in (X, τ), and each of which is bg-closed in (X, σ). However, f is not pre-bg-closed as $\{c\}$ is closed and thus b-closed in (X, τ) but $\{c\}$ is not bg-closed in (X, σ) (observe that $\{c\} \subset \{c\}$ but $b\,\mathrm{Cl}(\{c\}) = X \not\subset \{c\}$).*

The proof of the following lemma follows using a standard techneque, and thus omitted.

Lemma 2.7. *A function $f : X \to Y$ is pre-bg-closed (resp. regular bg-closed) if and only if for each subset B of Y and each $U \in BO(X)$ (resp. $U \in BR(X)$) containing $f^{-1}(B)$, there exists a bg-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.*

Corollary 2.8. *If $f : X \to Y$ is pre-bg-closed (resp. regular bg-closed), then for each closed set K of Y and each $U \in BO(X)$ (resp. $U \in BR(X)$) containing $f^{-1}(K)$, there exists $V \in BO(Y)$ containing K such that $f^{-1}(V) \subset U$.*

Proof. Suppose that $f : X \to Y$ is pre-bg-closed (resp. regular bg-closed). Let K be any closed set of Y and $U \in BO(X)$ (resp. $U \in BR(X)$) containing $f^{-1}(K)$. By Lemma 2.7, there exists a bg-open set G of Y such that $K \subset G$ and $f^{-1}(G) \subset U$. Since K is closed, $K \subset b\,\mathrm{Int}(G)$. Put $V = b\,\mathrm{Int}(G)$. Then, $K \subset V \in BO(Y)$ and $f^{-1}(V) \subset U$. □

Theorem 2.9. *If $f : X \to Y$ is continuous pre-bg-closed, then $f(H)$ is bg-closed in Y for each bg-closed set H of X.*

Proof. Let H be any bg-closed set of X and V an open set of Y containing $f(H)$. Since $f^{-1}(V)$ is an open set of X containing H, $b\,\mathrm{Cl}(H) \subset f^{-1}(V)$ and hence $f(b\,\mathrm{Cl}(H)) \subset V$. Since f is pre bg-closed and $b\,\mathrm{Cl}(H) \in BC(X)$, we have $b\,\mathrm{Cl}(f(H)) \subset b\,\mathrm{Cl}(f(b\,\mathrm{Cl}(H))) \subset V$. Therefore, $f(H)$ is bg-closed in Y. □

Definition 2.10. *A function $f : X \to Y$ is said to be pre-bg-continuous if $f^{-1}(K)$ is bg-closed in X for every $K \in BC(Y)$.*

It is obvious that a function $f : X \to Y$ is pre-bg-continuous if and only if $f^{-1}(V)$ is bg-open in X for every $V \in BO(Y)$.

Theorem 2.11. *If $f : X \to Y$ is closed pre-bg-continuous, then $f^{-1}(K)$ is bg-closed in X for each bg-closed set K of Y.*

Proof. Let K be a bg-closed set of Y and U an open set of X containing $f^{-1}(K)$. Put $V = Y \backslash f(X \backslash U)$, then V is open in Y, $K \subset V$, and $f^{-1}(V) \subset U$. Therefore, we have $b\,\mathrm{Cl}(K) \subset V$ and hence $f^{-1}(K) \subset f^{-1}(b\,\mathrm{Cl}(K)) \subset f^{-1}(V) \subset U$. Since f is pre-bg-continuous, $f^{-1}(b\,\mathrm{Cl}(K))$ is bg-closed in X and hence $b\,\mathrm{Cl}(f^{-1}(K)) \subset b\,\mathrm{Cl}(f^{-1}(b\,\mathrm{Cl}(K))) \subset U$. This shows that $f^{-1}(K)$ is bg-closed in X. □

Recall that a function $f : X \to Y$ is said to be b-irresolute [7] if $f^{-1}(V) \in BO(X)$ for every $V \in BO(Y)$.

Corollary 2.12. *If $f : X \to Y$ is closed b-irresolute, then $f^{-1}(K)$ is bg-closed in X for each bg-closed set K of Y.*

Corollary 2.13. *Let $f : X \to Y$ be a closed open continuous function. If K is a bg-closed set of Y, then $f^{-1}(K)$ is bg-closed in X.*

Proof. Follows from the fact that a continuous open function is b-irresolute. □

For the composition of pre-bg-closed functions, we have the following theorems.

Theorem 2.14. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the composition $g \circ f : X \to Z$ is pre-bg-closed if f is pre-bg-closed and g is continuous pre-bg-closed.*

Proof. The proof follows immediately from Theorem 2.9. □

Theorem 2.15. *Let $f : X \to Y$ and $g : Y \to Z$ be functions and let the composition $g \circ f \colon X \to Z$ be pre-bg-closed. Then the following hold:*

(i) *If f is a b-irresolute surjection, then g is pre-bg-closed;*
(ii) *If g is a closed pre-bg-continuous injection, then f is pre-bg-closed.*

Proof. (i) Let $K \in BC(Y)$. Since f is b-irresolute and surjective, $f^{-1}(K) \in BC(X)$ and $(g \circ f)(f^{-1}(K)) = g(K)$. Therefore, $g(K)$ is bg-closed in Z and hence g is pre-bg-closed.
(ii) Let $H \in BC(X)$. Then $(g \circ f)(H)$ is bg-closed in Z and $g^{-1}((g \circ f)(H)) = f(H)$. By Theorem 2.11, $f(H)$ is bg-closed in Y and hence f is pre-bg-closed. □

Recall that a topological space X is said to be b-normal [11] if for every disjoint closed sets A and B of X, there exist disjoint sets $U, V \in BO(X)$ such that $A \subset U$ and $B \subset V$.
The following lemma is analogous to Lemma 2.7, the straightforward proof is omitted.

Lemma 2.16. *A function $f : X \to Y$ is almost bg-closed if and only if for each subset B of Y and each $U \in RO(X)$ containing $f^{-1}(B)$, there exists a bg-open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.*

Corollary 2.17. *If $f : X \to Y$ is almost bg-closed, then for each closed set K of Y and each $U \in RO(X)$ containing $f^{-1}(K)$, there exists $V \in BO(Y)$ such that $K \subset V$ and $f^{-1}(V) \subset U$.*

Proof. The proof is similar to that of Corollary 2.8. □

Theorem 2.18. *Let $f : X \to Y$ be a continuous almost bg-closed surjection. If X is normal, then Y is b-normal.*

Proof. Let K_1 and K_2 be any disjoint closed sets of Y. Since f is continuous, $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are disjoint closed sets of X. By the normality of X, there exist disjoint open sets U_1 and U_2 such that $f^{-1}(K_i) \subset U_i$, where $i = 1, 2$. Now, put $G_i = \mathrm{Int}(\mathrm{Cl}(U_i))$ for $i = 1, 2$, then $G_i \in RO(X)$, $f^{-1}(K_i) \subset U_i \subset G_i$ and $G_1 \cap G_2 = \phi$. By Corollary 2.17, there exists $V_i \in BO(Y)$ such that $K_i \subset V_i$ and $f^{-1}(V_i) \subset G_i, i = 1, 2$. Since $G_1 \cap G_2 = \phi$, f is surjective we have $V_1 \cap V_2 = \phi$. This shows that Y is b-normal. □

Definition 2.19. [7] *A function $f : X \to Y$ is said to be b-open (resp. b-closed), if $f(U) \in BO(Y)$ (resp. $f(U) \in BC(X)$) for every open (resp. closed) set U of X.*

The following two corollaries are immediate consequences of Theorem 2.18.

Corollary 2.20. *If $f : X \to Y$ is a continuous b-closed surjection and X is normal, then Y is b-normal.*

Corollary 2.21. *If $f : X \to Y$ is a continuous and closed surjection and X is normal, then Y is b-normal.*

Definition 2.22. *A function $f : X \to Y$ is said to be pre-b-closed (resp. pre-b-open) if for each $F \in BC(X)$ (resp. $F \in BO(X)$), $f(F) \in BC(Y)$ (resp. $f(F) \in BO(Y)$).*

Remark 2.23. *It is clear that every pre-b-closed function is b-closed. The converse is, however, not true as the function f of Example 2.4 tells (observe that f is b-closed as $\{a\}$ and $\{b, c\}$ are b-closed in $(X, \dot{\sigma})$. However, f is not pre-b-closed as $\{a, b\}$ is b-closed in $(X, \dot{\tau})$ but not b-closed in $(X, \dot{\sigma})$.*

Theorem 2.24. [11] *Let A be a subset of a topological space X. Then*

(i) *$A \in BO(X)$ if and only if $b\,\mathrm{Cl}(A) \in BR(X)$.*
(ii) *$A \in BC(X)$ if and only if $b\,\mathrm{Int}(A) \in BR(X)$.*

Theorem 2.25. *Let $f : X \to Y$ be a continuous regular bg-closed surjection. If X is b-normal, then Y is b-normal.*

Proof. Although the proof is similar to that of Theorem 2.18, we will state it for the convenience of the reader. Let K_1 and K_2 be any disjoint closed sets of Y. Since f is continuous, $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are disjoint closed sets of X. By the b-normality of X, there exist disjoint sets $U_1, U_2 \in BO(X)$ such that $f^{-1}(K_i) \subset U_i$ for $i = 1, 2$. Now, put $G_i = b\,\mathrm{Cl}(U_i)$ for $i = 1, 2$, then by Theorem 2.24 $G_i \in BR(X), f^{-1}(K_i) \subset U_i \subset G_i$ and $G_1 \cap G_2 = \phi$. By Corollary 2.8, there exists $V_i \in BO(Y)$ such that $K_i \subset V_i$ and $f^{-1}(V_i) \subset G_i$, where $i = 1, 2$. Since f is surjective and $G_1 \cap G_2 = \phi$, we obtain $V_1 \cap V_2 = \phi$. This shows that Y is b-normal. □

Corollary 2.26. *Let $f : X \to Y$ be a continuous pre-bg-closed surjection. If X is b-normal, then Y is b-normal.*

Corollary 2.27. *If $f : X \to Y$ is a continuous pre-b-closed surjection and X is b-normal, then Y is b-normal.*

Theorem 2.28. *Let $f : X \to Y$ be a closed pre-bg-continuous injection. If Y is b-normal, then X is b-normal.*

Proof. Let H_1 and H_2 be disjoint closed sets of X. Since f is a closed injection, $f(H_1)$ and $f(H_2)$ are disjoint closed sets of Y. By the b-normality of Y, there exist disjoint sets $V_1, V_2 \in BO(Y)$ such that $f(H_i) \subset V_i$ for $i = 1, 2$. Since f

is pre-bg-continuous $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint bg-open sets of X and $H_i \subset f^{-1}(V_i)$ for $i = 1, 2$. Now, put $U_i = b\,\mathrm{Int}(f^{-1}(V_i))$ for $i = 1, 2$. Then, $U_i \in BO(X), H_i \subset U_i$ and $U_1 \cap U_2 = \phi$. This shows that X is b-normal. □

Corollary 2.29. *If $f : X \to Y$ is a closed b-irresolute injection and Y is b-normal, then X is b-normal.*

Proof. This is an immediate consequence of Theorem 2.28, since every b-irresolute function is pre bg-continuous. □

Recall that a topological space X is said to be b-regular [11] if for each closed set F and each point $x \in X \backslash F$, there exist disjoint $U, V \in BO(X)$ such that $x \in U$ and $F \subset V$.

Theorem 2.30. [11] *For a topological space X, the following properties are equivalent:*

(i) *X is b-regular;*
(ii) *For each U open in X and each $x \in U$, there exists $V \in BO(X)$ such that $x \in V \subset b\,\mathrm{Cl}(V) \subset U$;*
(iii) *For each U open in X and each $x \in U$, there exists $V \in BR(X)$ such that $x \in V \subset U$.*

Theorem 2.31. *Let $f : X \to Y$ be a continuous b-open almost bg-closed surjection. If X is regular, then Y is b-regular.*

Proof. Let $y \in Y$ and V be an open neighbourhood of y. Take a point $x \in f^{-1}(y)$. Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is open in X. By the regularity of X, there exists an open set U of X such that $x \in U \subset \mathrm{Cl}(U) \subset f^{-1}(V)$. Then $y \in f(U) \subset f(\mathrm{Cl}(U)) \subset V, f(U) \in BO(Y)$ and $f(\mathrm{Cl}(U))$ is bg-closed in Y. Therefore, we obtain $y \in f(U) \subset b\,\mathrm{Cl}(f(U)) \subset b\,\mathrm{Cl}(f(\mathrm{Cl}(U))) \subset V$. It follows from Theorem 2.30 that Y is b-regular. □

Corollary 2.32. *If $f : X \to Y$ is a continuous b-open b-closed surjection and X is regular, then Y is b-regular.*

Corollary 2.33. *If $f : X \to Y$ is a continuous b-open bg-closed surjection and X is regular, then Y is b-regular.*

Theorem 2.34. *Let $f : X \to Y$ be a continuous pre-b-open regular bg-closed surjection. If X is b-regular, then Y is b-regular.*

Proof. Let F be any closed set of Y and $y \in Y \backslash F$. Then $f^{-1}(F)$ is closed in X and $f^{-1}(F) \cap f^{-1}(y) = \phi$. Take a point $x \in f^{-1}(y)$. Since X is b-regular, there exist disjoint sets $U_1, U_2 \in BO(X)$ such that $x \in U_1$ and $f^{-1}(F) \subset U_2$. Therefore, we have $f^{-1}(F) \subset U_2 \subset b\,\mathrm{Cl}(U_2)$, $b\,\mathrm{Cl}(U_2) \in BR(X)$ and

$U_1 \cap b\,\mathrm{Cl}(U_2) = \phi$. Since f is regular bg-closed, by Corollary 2.8, there exists $V \in BO(Y)$ such that $F \subset V$ and $f^{-1}(V) \subset b\,\mathrm{Cl}(U_2)$. Since f is pre-b-open, we have $f(U_1) \in BO(Y)$. Moreoever, $U_1 \cap f^{-1}(V) = \phi$ and hence $f(U_1) \cap V = \phi$. Consequently, we obtain $y \in f(U_1) \in BO(Y)$, $F \subset V \in BO(Y)$ and $f(U_1) \cap V = \phi$. This shows that Y is b-regular. $\square$

Corollary 2.35. *If $f : X \to Y$ is a continuous pre-b-open pre-b-closed surjection and X is b-regular, then Y is b-regular.*

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