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On Upper and Lower Faintly γ-Continuous Multifunctions

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Abstract. The aim of this paper is to introduce and study upper and lower faintly γ-continuous multifunctions as a generalization of upper and lower γ-continuous multifunctions, respectively, due to Monsef and Nasef [1].

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1. Introduction

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions. This implies that both functions and multifunctions are important tools for studying properties of spaces and for constructing new spaces from previously existing ones. In 2001, Monsef and Nasef introduced the concept of γ-continuous multifunctions [1] in topological spaces. In this paper, we introduce and study upper and lower faintly γ-continuous multifunctions in topological spaces. We obtain some characterizations of these multifunctions and present several of their properties.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) mean topological spaces on which no separation axioms are assumed unless explicitly stated. For any subset A of X, the closure and the interior of A are denoted by $cl\,(A)$ and $int\,(A)$, respectively. A point $x \in X$ is called a θ-cluster point of A if $cl\,(V) \cap A \neq \phi$ for every open subset V of X containing x. The set of all θ-cluster points of A is called the θ-closure of A and is denoted by $cl_\theta\,(A)$. If $A = cl_\theta\,(A)$, then A is said to be θ-closed [8]. The complement of a θ-closed set is said to be θ-open. Clearly, A is θ-open if and only if for each $x \in A$, there exists an open set U such that $x \in U \subset cl\,(U) \subset A$. A subset A of (X, τ) is said to be γ-open [3] (resp. α-open [5]) if $A \subset int\,(cl\,(A)) \cup cl\,(int\,(A))$ (resp. $A \subset int\,(cl\,(int\,(A)))$). The complement of a γ-open set is called γ-closed. The family of all γ-open (resp. α-open) subsets of (X, τ) will be denoted by $\gamma O\,(X)$ (resp. $\alpha O\,(X)$). By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, also we always assume that $F\,(x) \neq \phi$ for all $x \in X$. For a multifunction $F : X \to Y$, the upper and lower inverse of any subset A of Y are denoted by $F^+\,(A)$ and $F^-\,(A)$, respectively, where $F^+\,(A) = \{x \in X : F\,(x) \subset A\}$ and $F^-\,(A) = \{x \in X : F\,(x) \cap A \neq \phi\}$. In particular, $F^-\,(y) = \{x \in X : y \in F\,(x)\}$ for each point $y \in Y$. A multifunction $F : X \to Y$ is said to be surjective if $F(X) = Y$. A multifunction

$F : (X, \tau) \to (Y, \sigma)$ is said to be lower γ-continuous [1] (resp. upper γ-continuous) multifunction if $F^{-}(V) \in \gamma O(X)$ (resp. $F^{+}(V) \in \gamma O(X)$) for every $V \in \sigma$.

3. faintly γ-continuous Multifunctions

Definition 3.1. *A multifunction $F : X \to Y$ is said to be:*
(i) upper faintly γ-continuous at $x \in X$ if for each θ-open subset V of Y containing $F(x)$, there exists $U \in \gamma O(X)$ containing x such that $F(U) \subset V$;
(ii) lower faintly γ-continuous at $x \in X$ if for each θ-open subset V of Y such that $F(x) \cap V \neq \phi$, there exists $U \in \gamma O(X)$ containing x such that $F(u) \cap V \neq \phi$ for every $u \in U$;
(iii) upper (resp. lower) faintly γ-continuous if it upper (resp. lower) faintly γ-continuous at each point of X.

Remark 3.2. *Since every θ-open set is open, it is clear that every upper (lower) γ-continuous multifunction is upper (lower) faintly γ-continuous. However, the converse is not true as the following simple example shows.*

Example 3.3. *Let $X = \{a, b, c\}, \tau = \{X, \phi, \{b\}\}, \sigma = \{X, \phi, \{a\}\}$. Then the multifunction $F : (X, \tau) \to (X, \sigma)$ defined by $F(x) = \{x\}$ is upper (lower) faintly γ-continuous (observe that the only θ-open subsets of (X, σ) are X and ϕ, so we may take $U = X$ in the definition of an upper (lower) faintly γ-continuous multifunction). However, F is not upper (lower) γ-continuous (observe that $F^{+}(\{a\}) = F^{-}(\{a\}) = \{a\}$ is not γ-open in (X, τ)).*

Theorem 3.4. *For a multifunction $F : X \to Y$, the following are equivalent:*
(i) F is upper faintly γ-continuous;
(ii) For each $x \in X$ and for each θ-open set V such that $x \in F^{+}(V)$, there exists a γ-open set U containing x such that $U \subset F^{+}(V)$;
(iii) For each $x \in X$ and for each θ-closed set V such that $x \in F^{+}(Y \backslash V)$, there exists a γ-closed set H such that $x \in X \backslash H$ and $F^{-}(V) \subset H$;
(iv) $F^{+}(V)$ is γ-open for any θ-open subset V of Y;
(v) $F^{-}(V)$ is γ-closed for any θ-closed subset V of Y;
(vi) $F^{-}(Y \backslash V)$ is γ-closed for any θ-open subset V of Y;
(vii) $F^{+}(Y \backslash V)$ is γ-open for any θ-closed subset V of Y.

Proof. (i)$\leftrightarrow$(ii): Clear.
(ii)$\leftrightarrow$(iii): Let $x \in X$ and V be a θ-closed subset of Y such that $x \in F^{+}(Y \backslash V)$. By (ii), there exists a γ-open set U containing x such that $U \subset F^{+}(Y \backslash V)$.

Thus $F^-(V) \subset X\backslash U$. Take $H = X\backslash U$. Then $x \in X\backslash H$ and H is γ-closed. The converse is similar.

(i)$\leftrightarrow$(iv): Let $x \in F^+(V)$ and V be a θ-open subset of Y. By (i), there exists a γ-open set U_x containing x such that $U_x \subset F^+(V)$. Thus, $F^+(V) = \cup_{x\in F^+(V)} U_x$. Since any union of γ-open sets is γ-open, $F^+(V)$ is γ-open. The converse is clear.

(iv)$\leftrightarrow$(vii) and (v)$\leftrightarrow$(vi): Clear.

(iv)$\leftrightarrow$(vi): Follows from the fact that $F^-(V) = X\backslash F^+(Y\backslash V)$. □

Theorem 3.5. *For a multifunction $F : X \to Y$, the following are equivalent:*
(i) F is lower faintly γ-continuous;
(ii) For each $x \in X$ and for each θ-open set V such that $x \in F^-(V)$, there exists a γ-open set U containing x such that $U \subset F^-(V)$;
(iii) For each $x \in X$ and for each θ-closed set V such that $x \in F^-(Y\backslash V)$, there exists a γ-closed set H such that $x \in X\backslash H$ and $F^+(V) \subset H$;
(iv) $F^-(V)$ is γ-open for any θ-open subset V of Y;
(v) $F^+(V)$ is γ-closed for any θ-closed subset V of Y;
(vi) $F^+(Y\backslash V)$ is γ-closed for any θ-open subset V of Y;
(vii) $F^-(Y\backslash V)$ is γ-open for any θ-closed subset V of Y.

Proof. Similar to that of Theorem 3.4. □

Lemma 3.6. [3] *If $A \in \alpha O(X)$ and $B \in \gamma O(X)$, then $A \cap B \in \gamma O(A)$.*

Theorem 3.7. *Let $F : X \to Y$ be a multifunction and $U \in \alpha O(X)$. If F is lower (upper) faintly γ-continuous, then the multifunction $F_{|U} : U \to Y$ is lower (upper) faintly γ-continuous.*

Proof. Let V be any θ-open subset of Y . Then $F^-_{|U}(V) = F^-(V) \cap U$ and $F^+_{|U}(V) = F^+(V) \cap U$. Thus the result follows from Theorems 3.4, 3.5 and Lemma 3.6. □

Theorem 3.8. *Suppose that (X, τ) and (X_α, τ_α) are topological spaces where $\alpha \in J$. Let $F : X \to \Pi_{\alpha\in J} X_\alpha$ be a multifunction from X to the product space $\Pi_{\alpha\in J} X_\alpha$ and let $P_\alpha : \Pi_{\alpha\in J} X_\alpha \to X_\alpha$ be the projection multifunction for each $\alpha \in J$ which is defined by $P_\alpha((x_\alpha)) = \{x_\alpha\}$. If F is upper (lower) faintly γ-continuous, then $P_\alpha \circ F$ is upper (lower) faintly γ-continuous for each $\alpha \in J$.*

Proof. Let V_α be a θ-open set in (X_α, τ_α). Then $(P_\alpha \circ F)^+(V_\alpha) = F^+(P^+_\alpha(V_\alpha)) = F^+(V_\alpha \times \Pi_{\beta\neq\alpha} X_\beta)$ (resp. $(P_\alpha \circ F)^-(V_\alpha) = F^-(P^-_\alpha(V_\alpha)) = F^-(V_\alpha \times \Pi_{\beta\neq\alpha} X_\beta)$).

Since F is upper (lower) faintly γ-continuous and since $V_\alpha \times \Pi_{\beta\neq\alpha}X_\beta$ is a θ-open set, it follows from Theorems 3.4 and 3.5 that $F^+(V_\alpha \times \Pi_{\beta\neq\alpha}X_\beta)$ (resp. $F^-(V_\alpha \times \Pi_{\beta\neq\alpha}X_\beta)$) is a γ-open set in (X,τ). Hence again by Theorems 3.4 and 3.5, $P_\alpha \circ F$ is upper (lower) faintly γ-continuous for each $\alpha \in J$. □

Corollary 3.9. *Let $F : X \to Y$ be a multifunction. If the graph multifunction G_F of F is upper (lower) faintly γ-continuous, then F is upper (lower) faintly γ-continuous, where $G_F : X \to X \times Y, G_F(x) = \{x\} \times F(x)$.*

Corollary 3.10. *Suppose that $(X,\tau),(Y,\sigma),(Z,\eta)$ are topological spacesand $F_1 : X \to Y, F_2 : X \to Z$ are multifunctions. Let $F_1 \times F_2 : X \to Y \times Z$ be the multifunction defined by $F_1 \times F_2(x) = F_1(x) \times F_2(x)$ for each $x \in X$. If $F_1 \times F_2$ is upper (lower) faintly γ-continuous, then F_1 and F_2 are upper (lower) faintly γ-continuous.*

The following lemma can be easily established.

Lemma 3.11. *If $A \times B \in \gamma O(X \times Y)$, then $A \in \gamma O(X)$ and $B \in \gamma O(Y)$.*

Theorem 3.12. *Suppose that (X_α,τ_α) and (Y_α,σ_α) are topological spaces for each $\alpha \in J$. Let $F_\alpha : X_\alpha \to Y_\alpha$ be a multifunction for each $\alpha \in J$ and let $F : \Pi_{\alpha\in J}X_\alpha \to \Pi_{\alpha\in J}Y_\alpha$ be the multifunction defined by $F((x_\alpha)) = \Pi_{\alpha\in J}F_\alpha(x_\alpha)$. If F is upper (lower) faintly γ-continuous, then F_α is upper (lower) faintly γ-continuous for each $\alpha \in J$.*

Proof. Let V_α be a θ-open subset of Y_α. Then $V_\alpha \times \Pi_{\beta\neq\alpha}Y_\beta$ is a θ-open set. Since F is upper (lower) faintly γ-continuous, it follows from Theorems 3.4 and 3.5 that $F^+(V_\alpha \times \Pi_{\beta\neq\alpha}Y_\beta) = F_\alpha^+(V_\alpha) \times \Pi_{\beta\neq\alpha}X_\beta$ (resp. $F^-(V_\alpha \times \Pi_{\beta\neq\alpha}Y_\beta) = F_\alpha^-(V_\alpha) \times \Pi_{\beta\neq\alpha}X_\beta$). Consequently, it follows from Lemma 3.11 that $F_\alpha^+(V_\alpha)$ (resp. $F_\alpha^-(V_\alpha)$) is a γ-open set. Thus again by Theorems 3.4 and 3.5, F_α is upper (lower) faintly γ-continuous for each $\alpha \in J$. □

Corollary 3.13. *Suppose that $F_1 : X_1 \to Y_1, F_2 : X_2 \to Y_2$ are multifunctions. If $F_1 \times F_2$ is upper (lower) faintly γ-continuous, then F_1 and F_2 are upper (lower) faintly γ-continuous, where $F_1 \times F_2$ is the product multifunction defined as follows: $F_1 \times F_2 : X_1 \times X_2 \to Y_1 \times Y_2, F_1 \times F_2((x_1,x_2)) = F_1(x_1) \times F_2(x_2)$, where $x_1 \in X_1$ and $x_2 \in X_2$.*

Lemma 3.14. [4] *If $A \in \gamma O(X)$ and $B \in \gamma O(Y)$, then $A \times B \in \gamma O(X \times Y)$.*

Recall that a multifunction $F : X \to Y$ is said to be punctually closed if for each $x \in X$, $F(x)$ is closed. Recall also that a space X is called θ-normal if for any disjoint closed subsets F_1, F_2 of X, there exist two disjoint θ-open subsets V_1, V_2 of X containing F_1, F_2, respectively.

Theorem 3.15. *If Y is a θ-normal space and $F_i : X_i \to Y$ is an upper faintly γ-continuous multifunction such that F_i is punctually closed for $i = 1, 2$, then the set $\{(x_1, x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \phi\}$ is γ-closed.*

Proof. Let $A = \{(x_1, x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \phi\}$ and $(x_1, x_2) \in (X_1 \times X_2) \backslash A$. Then $F_1(x_1) \cap F_2(x_2) = \phi$. Since Y is θ-normal and F_i is punctually closed for $i = 1, 2$, there exist disjoint θ-open sets V_1, V_2 such that $F_i(x_i) \subset V_i$ for $i = 1, 2$. Since F_i is upper faintly γ-continuous, it follows from Theorem 3.4 that $F_i^+(V_i)$ is a γ-open set in X_i for $i = 1, 2$. Put $U = F_1^+(V_1) \cap F_2^+(V_2)$. Then by Lemma 3.14, U is γ-open, also $(x_1, x_2) \in U \subset (X_1 \times X_2) \backslash A$. Hence, $(X_1 \times X_2) \backslash A$ is γ-open, that is, A is γ-closed. □

Definition 3.16. *A topological space (X, τ) is said to be γ-T_2* [2] *(resp. θ-T_2* [7]*) if for each pair of distinct points x and y of X, there exist disjoint γ-open (resp. θ-open) subsets U and V of X containing x and y, respectively.*

Theorem 3.17. *Let $F : X \to Y$ be an upper faintly γ-continuous multifunction and punctually closed from a topological space X into a θ-normal space Y such that $F(x) \cap F(y) = \phi$ for each pair of distinct points x and y of X. Then X is γ-T_2.*

Proof. Let x and y be any two distinct points of X. Then $F(x) \cap F(y) = \phi$. Since Y is θ-normal and F is punctually closed, there exist disjoint θ-open sets U and V containing $F(x)$ and $F(y)$, respectively, but F is upper faintly γ-continuous, so it follows from Theorem 3.4 that $F^+(U)$ and $F^+(V)$ are disjoint γ-open subsets of X containing x and y, respectively. Hence, X is γ-T_2. □

Definition 3.18. *A topological space (X, τ) is said to be θ-compact* [7] *(resp. γ-compact* [3]*) if every θ-open (resp. γ-open) cover of X has a finite subcover. A subset A of a topological space X is said to be θ-compact relative to X if every cover of A by θ-open subsets of X has a finite subcover of A.*

Theorem 3.19. *Let $F : X \to Y$ be an upper faintly γ-continuous surjective multifunction such that $F(x)$ is θ-compact relative to Y for each $x \in X$. If X is γ-compact, then Y is θ-compact.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a θ-open cover of Y. Since $F(x)$ is θ-compact relative to Y for each $x \in X$, there exists a finite subset $\Lambda(x)$ of Λ such that

$F(x) \subset \cup_{\alpha\in\Lambda(x)} V_\alpha$. Put $V(x) = \cup_{\alpha\in\Lambda(x)} V_\alpha$. Then $V(x)$ is a θ-open subset of Y containing $F(x)$. Since F is upper faintly γ-continuous, it follows from Theorem 3.4 that $F^+(V(x))$ is a γ-open subset of X containing $\{x\}$. Thus the family $\{F^+(V(x)) : x \in X\}$ is a γ-open cover of X, but X is γ-compact, so there exist $x_1, x_2, ..., x_n \in X$ such that $X = \cup_{i=1}^n F^+(V(x_i))$. Hence, $Y = F\left(\cup_{i=1}^n F^+(V(x_i))\right) = \cup_{i=1}^n F\left(F^+(V(x_i))\right) \subset \cup_{i=1}^n V(x_i) = \cup_{i=1}^n \cup_{\alpha\in\Lambda(x_i)} V_\alpha$. Hence, Y is θ-compact. □

Definition 3.20. *Let $F : X \to Y$ be a multifunction. The multigraph $G(F) = \{(x, y) : y \in F(x), x \in X\}$ of F is said to be γ-θ-closed if for each $(x, y) \in (X \times Y) \backslash G(F)$, there exist a γ-open set U and a θ-open set V containing x and y, respectively, such that $(U \times V) \cap G(F) = \phi$, i.e. $F(U) \cap V = \phi$.*

Theorem 3.21. *If a multifunction $F : X \to Y$ is upper faintly γ-continuous such that $F(x)$ is θ-compact relative to Y for each $x \in X$ and Y is θ-T_2, then the multigraph $G(F)$ of F is γ-θ-closed.*

Proof. Let $(x, y) \in (X \times Y) \backslash G(F)$. Then $y \in Y \backslash F(x)$. Since Y is θ-T_2, for each $z \in F(x)$, there exist disjoint θ-open subsets $U(z)$ and $V(z)$ of Y containing z and y, respectively. Thus $\{U(z) : z \in F(x)\}$ is a θ-open cover of $F(x)$, but $F(x)$ is θ-compact relative to Y, so there exist $z_1, z_2, ..., z_n \in F(x)$ such that $F(x) \subset \cup_{i=1}^n U(z_i)$. Put $U = \cup_{i=1}^n U(z_i)$ and $V = \cap_{i=1}^n V(z_i)$. Then U and V are θ-open subsets of Y such that $F(x) \subset U, y \in V$ and $U \cap V = \phi$. Since F is upper faintly γ-continuous, it follows from Theorem 3.4 that $F^+(U)$ is a γ-open subset of X. Also $x \in F^+(U)$ since $F(x) \subset U$ and $F(F^+(U)) \cap V = \phi$ since $U \cap V = \phi$. Hence, $G(F)$ is γ-θ-closed. □

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