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Low Separation Axioms via $\widetilde{g}D$-sets

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Abstract. In this paper, we introduce three classes of topological spaces called $\widetilde{g}$-D_0, $\widetilde{g}$-D_1 and $\widetilde{g}$-D_2 spaces interms of the concept of $\widetilde{g}D$-sets and investigate some of their fundamental properties.

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1. Introduction

The notion of $\widetilde{g}$-open sets was introduced by Jafari et al [1] as a generalization of open sets. In this paper, we introduce three low separation axioms by utilizing $\widetilde{g}D$-sets. We characterize these spaces and study their fundamental properties. Throughout this paper, (X,τ) and (Y,σ) (or X and Y) stand for topological spaces. For any subset A of X, the closure and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

2. Preliminaries

Before entering to our work we recall the following definitions and results which are used in this paper.

Definition 2.1. *Let A be a subset of a topological space (X, τ). Then*

(1) *A is semi-open* [2], *if $A \subset \mathrm{Cl}(\mathrm{Int}\, A)$. The complement of a semi-open set is called semi-closed, and the semi-closure of A, denoted by $\mathrm{scl}\,(A)$, is the smallest semi-closed set containing A.*
(2) *A is $\widehat{g}$-closed* [5], *if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is semi-open in (X, τ). The complement of a $\widehat{g}$-closed set is called $\widehat{g}$-open.*
(3) *A is $^*g$-closed* [7] *if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is $\widehat{g}$-open in (X, τ). The complement of a $^*g$-closed set is called $^*g$-open.*
(4) *A is $^\#g$-semi-closed* [6], *if $\mathrm{scl}\,(A) \subset U$ whenever $A \subset U$ and U is $^*g$-open. The complement of a $^\#g$-semi-closed set is called $^\#g$-semi-open.*
(5) *A is $\widetilde{g}$-closed* [1], *if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is $^\#g$-semi-open. The complement of a $\widetilde{g}$-closed set is called $\widetilde{g}$-open. The class of all $\widetilde{g}$-open subsets of (X, τ) is denoted by $\widetilde{g}O(X, \tau)$.*

Theorem 2.2. [1] *In any space X, the following hold:*

(1) *An arbitary intersection of $\widetilde{g}$-closed sets is $\widetilde{g}$-closed.*
(2) *The finite union of $\widetilde{g}$-closed sets is $\widetilde{g}$-closed.*

Definition 2.3. [3] *A map $f : (X, \tau) \to (Y, \sigma)$ is said to be $\widetilde{g}$-irresolute if the inverse image of every $\widetilde{g}$-open set in Y is $\widetilde{g}$-open in X.*

Definition 2.4. [4] *A subset B_x of a topological space X is said to be a $\widetilde{g}$-neighbourhood of a point $x \in X$ if there exists a $\widetilde{g}$-open set U such that $x \in U \subset B_x$.*

Definition 2.5. [4] *A topological space (X, τ) is said to be:*

(1) *$\widetilde{g}$-T_0, if for any distinct pair of points in X, there is a $\widetilde{g}$-open set containing one of the points but not the other.*
(2) *$\widetilde{g}$-T_1, if for any distinct pair of points x and y in X, there is a $\widetilde{g}$-open set U in X containing x but not y and a $\widetilde{g}$-open set V in X containing y but not x.*
(3) *$\widetilde{g}$-T_2, if for any distinct pair of points x and y in X, there exist $\widetilde{g}$-open sets U and V in X containing x and y, respectively, such that $U \cap V = \phi$.*

Theorem 2.6. [4] *For a space X, the following statements are equivalent:*

(1) *X is $\widetilde{g}$-T_1;*
(2) *Every singleton of X is $\widetilde{g}$-closed in X.*

3. $\widetilde{g}D$-sets and associated separation axioms

Definition 3.1. *A subset A of a topological space X is called a $\widetilde{g}D$-set if there are two sets $U, V \in \widetilde{g}O(X, \tau)$ such that $U \neq X$ and $A = U \backslash V$.*

Remark 3.2. *Letting $A = U$ and $V = \phi$ in Definition 3.1, it follows that every proper $\widetilde{g}$-open subset U of a space X is a $\widetilde{g}D$-set.*

Definition 3.3. *A topological space (X, τ) is called $\widetilde{g}$-D_0 if for any distinct pair of points x and y of X there exists a $\widetilde{g}D$-set of X containing x but not y or a $\widetilde{g}D$-set of X containing y but not x.*

Definition 3.4. *A topological space (X, τ) is called $\widetilde{g}$-D_1 if for any distinct pair of points x and y of X, there exists a $\widetilde{g}D$-set of X containing x but not y and a $\widetilde{g}D$-set of X containing y but not x.*

Definition 3.5. *A topological space (X, τ) is called $\widetilde{g}$-D_2 if for any distinct pair of points x and y of X there exist disjoint $\widetilde{g}D$-sets G and E of X containing x and y, respectively.*

Proposition 3.6. *(1) If (X, τ) is $\widetilde{g}$-T_i, then (X, τ) is $\widetilde{g}$-$D_i, i = 0, 1, 2$.*
(2) If (X, τ) is $\widetilde{g}$-D_i, then it is $\widetilde{g}$-$D_{i-1}, i = 1, 2$.

Proof. (1) Follows from Remark 3.2.
(2) Clear. □

Theorem 3.7. *For a topological space (X, τ), the following statements hold:*

(1) *(X, τ) is $\widetilde{g}$-D_0 if and only if it is $\widetilde{g}$-T_0.*
(2) *(X, τ) is $\widetilde{g}$-D_1 if and only if it is $\widetilde{g}$-D_2.*

Proof. (1) **Sufficiency.** Follows from Proposition 3.6 (1).
Necessity. Let (X, τ) be $\widetilde{g}$-D_0. Then for each distinct pair $x, y \in X$, at least one of x, y say x belongs to a $\widetilde{g}D$-set G where $y \notin G$. Let $G = U_1 \backslash U_2$ such that $U_1 \neq X$ and $U_1, U_2 \in \widetilde{g}O(X, \tau)$. Then $x \in U_1$. For $y \notin G$, we have two cases:
(a) $y \notin U_1$; (b) $y \in U_1$ and $y \in U_2$.
In case (a), $x \in U_1$ but $y \notin U_1$;
In case (b), $y \in U_2$ but $x \notin U_2$. Hence, X is $\widetilde{g}$-T_0.
(2) **Sufficiency.** Follows from Proposition 3.6 (2).
Necessity. Suppose that X is $\widetilde{g}$-D_1. Then for each distinct pair $x, y \in X$, we have $\widetilde{g}D$-sets G_1, G_2 such that $x \in G_1, y \notin G_1$; $y \in G_2, x \notin G_2$. Let $G_1 = U_1 \backslash U_2, G_2 = U_3 \backslash U_4$, where $U_1, U_2, U_3, U_4 \in \widetilde{g}O(X, \tau)$ and $U_1 \neq X, U_3 \neq X$. By $x \notin G_2$, it follows that either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. Now we consider two cases:
(i) $x \notin U_3$. By $y \notin G_1$ we have two subcases:
(a) $y \notin U_1$. By $x \in U_1 \backslash U_2$, it follows that $x \in U_1 \backslash (U_2 \cup U_3)$ and by $y \in U_3 \backslash U_4$ we have $y \in U_3 \backslash (U_2 \cup U_4)$. Also, $(U_1 \backslash (U_2 \cup U_3)) \cap (U_3 \backslash (U_1 \cup U_4)) = \phi$. Observe also from Theorem 2.2 (1) that $U_2 \cup U_3$ and $U_1 \cup U_4$ are $\widetilde{g}$-open.

(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 \backslash U_2, y \in U_2$ and $(U_1 \backslash U_2) \cap U_2 = \phi$. Observe that $U_2 \neq X$ since $G_1 \neq \phi$, thus by Remark 3.2, U_2 is a $\widetilde{g}D$-set.
(ii) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 \backslash U_4, x \in U_4$ and $(U_3 \backslash U_4) \cap U_4 = \phi$. Observe that $U_4 \neq X$ since $G_2 \neq \phi$, thus by Remark 3.2, U_4 is a $\widetilde{g}D$-set.
Hence, X is $\widetilde{g}$-D_2. □

Corollary 3.8. *If (X, τ) is $\widetilde{g}$-D_1, then it is $\widetilde{g}$-T_0.*

Proof. Follows from Proposition 3.6 (2) and Theorem 3.7 (1). □

Definition 3.9. *A point $x \in X$ which has X as the only $\widetilde{g}$-neighbourhood is called a $\widetilde{g}$-neat point.*

Theorem 3.10. *For a $\widetilde{g}$-T_0 topological space (X, τ) whose cardinality is greater than 1, the following are equivalent:*

(1) *(X, τ) is $\widetilde{g}$-D_1;*
(2) *(X, τ) has no $\widetilde{g}$-neat point.*

Proof. (1)→(2): Since (X, τ) is $\widetilde{g}$-D_1, each point x of X is contained in a $\widetilde{g}D$-set $O = U \backslash V$ and thus in U. By definition $U \neq X$. This implies that x is not a $\widetilde{g}$-neat point.
(2)→(1): Since X is $\widetilde{g}$-T_0, for each distinct pair of points $x, y \in X$, at least one of them, x (say) has a $\widetilde{g}$-neighbourhood U containing x and not y. Thus U is different from X, and therefore by Remark 3.2, U is a $\widetilde{g}D$-set. Since X has no $\widetilde{g}$-neat point, y is not a $\widetilde{g}$-neat point. Thus, there exists a $\widetilde{g}$-neighbourhood V of y such that $V \neq X$. Therefore, $y \in V \backslash U, x \notin V \backslash U$ and $V \backslash U$ is a $\widetilde{g}D$-set. Hence, X is $\widetilde{g}$-D_1. □

Remark 3.11. *It is clear that a $\widetilde{g}$-T_0 topological space (X, τ) is not $\widetilde{g}$-D_1 if and only if there is a unique $\widetilde{g}$-neat point in X. It is unique because if x and y are both $\widetilde{g}$-neat points in X, then at least one of them say x has a $\widetilde{g}$-neighbourhood U containing x but not y. But this is a contradiction since $U \neq X$.*

The following diagram summarizes the implications among $\widetilde{g}$-T_i spaces and $\widetilde{g}$-D_i spaces, $i = 0, 1, 2$.

$$\begin{array}{ccccc} \widetilde{g}\text{-}T_2 & \rightarrow & \widetilde{g}\text{-}T_1 & \rightarrow & \widetilde{g}\text{-}T_0 \\ \downarrow & & \downarrow & & \updownarrow \\ \widetilde{g}\text{-}D_2 & \leftrightarrow & \widetilde{g}\text{-}D_1 & \rightarrow & \widetilde{g}\text{-}D_0 \end{array}$$

It was shown in [4] that the implications in the first row of the above diagram are not reversible. The following examples, however, show that other implications are not reversible too.

Example 3.12. *Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Observing that the $\widetilde{g}$-open subsets of X are the open sets, it was pointed out* *in [4] that the space X is $\widetilde{g}$-T_0. Thus, X is $\widetilde{g}$-D_0. However, it follows from Theorem 3.10 that X is not $\widetilde{g}$-D_1 (observe that X has c as a $\widetilde{g}$-neat point).*

Example 3.13. *Let $X = \{a, b, c\}$ with topology $\tau = \{\phi, \{a\}, \{a, c\}, \{a, b\}, X\}$. Then the topological space (X, τ) is clearly $\widetilde{g}$-T_0 as it is T_0. Also, X has no $\widetilde{g}$-neat points as each of its points has a proper open and thus $\widetilde{g}$-open neighbourhood. Thus, it follows from Theorem 3.10 that X is $\widetilde{g}$-D_1. However, it follows from Theorem 2.6 that X is not $\widetilde{g}$-T_1 (observe that $\{a\}$ is not $\widetilde{g}$-closed since $\{a\} \subset \{a\}$, $\{a\}$ is open, thus semi-open and therefore ${}^{\#}g$-semi-open, but $\mathrm{Cl}(\{a\}) = X \not\subset \{a\}$).*

Remark 3.14. *Observe from the above diagram that any space which is $\widetilde{g}$-D_1 that is not $\widetilde{g}$-T_1 is $\widetilde{g}$-D_2 that is not $\widetilde{g}$-T_2.*

Theorem 3.15. *If $f : (X, \tau) \to (Y, \sigma)$ is a $\widetilde{g}$-irresolute surjective function and E is a $\widetilde{g}D$-set in Y, then the inverse image of E is a $\widetilde{g}D$-set in X.*

Proof. Let E be a $\widetilde{g}D$-set in Y. Then there are $\widetilde{g}$-open sets U_1 and U_2 in Y such that $E = U_1 \backslash U_2$ and $U_1 \neq Y$. By the $\widetilde{g}$-irresoluteness of f, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are $\widetilde{g}$-open in X. Since $U_1 \neq Y$ and f is surjective, $f^{-1}(U_1) \neq X$. Hence, $f^{-1}(E) = f^{-1}(U_1) \backslash f^{-1}(U_2)$ is a $\widetilde{g}D$-set. □

Theorem 3.16. *If (Y, σ) is $\widetilde{g}$-D_1 and $f : (X, \tau) \to (Y, \sigma)$ is $\widetilde{g}$-irresolute and bijective, then (X, τ) is $\widetilde{g}$-D_1.*

Proof. Suppose that Y is a $\widetilde{g}$-D_1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is $\widetilde{g}$-D_1, it follows from Theorem 3.7 (2) that Y is $\widetilde{g}$-D_2, thus there exist disjoint $\widetilde{g}D$-sets G_x and G_y of Y containing $f(x)$ and $f(y)$, respectively. By Theorem 3.15, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint $\widetilde{g}D$-sets in X containing x and y, respectively. Hence, X is $\widetilde{g}$-D_1. □

Theorem 3.17. *A topological space (X, τ) is $\widetilde{g}$-D_1 if and only if for each pair of distinct points $x, y \in X$, there exists a $\widetilde{g}$-irresolute surjective function $f : (X, \tau) \to (Y, \sigma)$, where (Y, σ) is a $\widetilde{g}$-D_1 space such that $f(x)$ and $f(y)$ are distinct.*

Proof. **Necessity.** For every pair of distinct points of X, it suffices to take the identity function on X.

Sufficiency. Let x and y be any pair of distinct points in X. By hypothesis, there exists a $\widetilde{g}$-irresolute function f from (X, τ) onto a $\widetilde{g}$-D_1 space (Y, σ) such that $f(x) \neq f(y)$. Thus by Theorem 3.7 (2), (Y, σ) is $\widetilde{g}$-D_2, and therefore there exist disjoint $\widetilde{g}D$-sets G_x and G_y in Y such that $f(x) \in G_x$ and $f(y) \in G_y$. Since f is $\widetilde{g}$-irresolute and surjective, by Theorem 3.15, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint $\widetilde{g}D$-sets in X containing x and y, respectively. Hence, X is $\widetilde{g}$-D_1. □

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