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Number Theory through Functional Analysis

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Abstract

Arithmetic functions, Multiplicative functions, Completely multiplicative functions and their analytical structures have been discussed in this paper.

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Introduction

This paper deals with the structure of the set of all arithmetic functions. Also the paper is devoted to show that the set of all non-zero multiplicative functions is a group.

§1. Arithmetic Functions

Notations 1.1:

$Z^{+} = \{all \quad positive \quad integers\}$
$C = \{all \quad complex \quad numbers\}$

$|Z^{+}|$ = The cardinality of Z^{+}.

Definition 1.2:

Any function $f : Z^{+} \to C$ is called a complex-valued arithmetic function

$A = \{\text{all arithmetic functions}\}$

The cardinality of A = $2^{|Z^{+}|}$

Definition 1.3: (Paranorm)

g is a paranorm on X if

(i) $g : X \to R$ with $g(x) \geq 0$

(ii) $g(x) = 0 \Leftrightarrow x = 0$

(iii) $g(x) = g(-x) \quad \forall x \in X$

(iv) $g(x+y) \leq g(x) + g(y) \quad \forall x, y \in X$

(v) If $\lambda_n, \lambda_0 \in C$ with $\lambda_n \to \lambda_0 \ (n \to \infty)$
and if $x_n, a \in X$ with $x_n \to a \ (n \to \infty)$ in the sense that

$g(x_n - a) \to 0 \ (n \to \infty)$, then $\lambda_n x_n \to \lambda_0 a \ (n \to \infty)$ in the sense that $g(\lambda_n x_n - \lambda_0 a) \to 0 \ (n \to \infty)$.

Note :

Define for each $f = \{f(k)\} \in A$,

$$p(f)=\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{|f(k)|}{1+|f(k)|}\le\sum_{k=1}^{\infty}\frac{1}{2^k}<\infty \qquad ...(1)$$

The p is a paranorm for A.

Proposition 1.4:

$$p(f_n)\to 0 \Leftrightarrow |f_n(k)|\to 0 \ \ \forall k \text{ as } (n\to\infty). \qquad ...(2)$$

Proof:

$$\frac{1}{2^k}\frac{|f(k)|}{1+|f(k)|}\le p(f) \Rightarrow \frac{|f(k)|}{1+|f(k)|}\le 2^k p(f)$$

Suppose that $f_n\to 0 \ \ \forall k$ as $n\to\infty$

$\Rightarrow \ 2^k p(f_n)\to 0$

$\Rightarrow \ |f_n(k)|\to 0 \ \ \forall k$ as $n\to\infty$.

Similarly the converse holds.

Note: With linear operations: (1)$(f+g)(k)=f(k)+g(k)$

(2)$(\alpha f)(k)=\alpha f(k)$

A is a linear space over C.

Theorem 1.5:

A is a metric linear space.

Proof:

Step 1:

Let $f, g \in A$. Define $d(f,g) = \sum_{k=1}^{\infty} \frac{f(k) - g(k)}{2^k \left(1 + \left| f(k) - g(k) \right|\right)}$.

Then d is a metric for A.

Note that $\left| f(k) - h(k) - g(k) + h(k) \right| = \left| f(k) - g(k) \right| \ \forall k \in Z^+$.

$\Rightarrow$ d is translation invariant.

Step 2:

Also, $d(x+y, a+b) \le d(x,a) + d(y,b)$.

$\Rightarrow$ Addition is continuous in (A, d).

Step 3:

Let $\lambda_0 \in C$ and $f \in A$. We shall show that $d\left(f^{(n)}, f\right) \to 0 \ (n \to \infty)$.
This is equivalent to $f^{(n)}(k) \to f(k) \ \left(n \to \infty, \ \forall k \in Z^+\right)$.
Then $\lambda_n \to \lambda_0$, $f^{(n)} \to f \ (n \to \infty)$.

$\Rightarrow$ $\lambda_n f^{(n)}(k) \to \lambda_0 f(k) \ \left(n \to \infty, \ \forall k \in Z^+\right)$

$\Rightarrow$ $\lambda_n f^n \to \lambda_0 f \ (n \to \infty)$

$\Rightarrow$ Scalar multiplication is continuous.

Step 4:

Suppose that $d\left(f^{(n)}, f\right) \to 0 \ (n \to \infty)$.

Write $b(n,k) = \left| f^n{}_k - f_k \right|$. Take any $k \in Z^+$.
Then $\varepsilon(n,k) = \frac{b(n,k)}{1 + b(n,k)} \le 2^k \cdot d\left(f^n, f\right) \to 0$ as $n \to \infty$.
But $0 \le \varepsilon(n,k) < 1$. Hence $b(n,k) = \frac{\varepsilon(n,k)}{1 - \varepsilon(n,k)}$.
$\Rightarrow$ $b(n,k) \to 0 \ (n \to \infty)$.

$\Rightarrow \ f^{n}{}_{k} \to f_k \ \ \left(n \to \infty, \ \forall k \in Z^+\right).$

Conversely, suppose that $f_k^{(n)} \to f_k$.

$\Rightarrow \ b(n,k) \to 0 \ \left(n \to \infty, \ \forall k \in Z^+\right)$

$\Rightarrow \ \forall n, m \in Z^+,$

$$d\left(f^{(n)}, f\right) = \sum_{k=1}^{m} \frac{b(n,k)}{2^k\left(1+b(n,k)\right)} + \sum_{k=m+1}^{\infty} \frac{b(n,k)}{2^k\left(1+b(n,k)\right)}$$

$$\leq \sum_{k=1}^{m} \frac{b(n,k)}{2^k\left(1+b(n,k)\right)} + \sum_{k=m+1}^{\infty} \frac{1}{2^k}$$

Let $\varepsilon > 0$. Choose $m = m(\varepsilon) \in Z^+$ such that the sum of the second series in equation (1) is less than $\varepsilon / 2$.

With this choice of m, the first sum in equation (5) $\to$ 0 as $n \to \infty$.

Since $b(n,k) \to 0 \ \left(n \to \infty, \ \forall k \in Z^+\right)$ by our hypothesis, we have

$d\left(f^{n}, f\right) < \varepsilon$ for sufficiently large n.

Hence A is a linear metric space.

§2. Multiplicative Functions

Definition 2.1:

Let $\gcd(m,n) = 1$ with $m, n \in Z^+$. A function $f \in A$ is called a multiplicative function if

$$f(mn) = f(m)\, f(n). \qquad ...(3)$$

Let $M = \{$all multiplicative functions$\}$. M is not a linear sub-space of A.

However, we have the following result.

Theorem 2.2:

M is a closed subset of A.

Proof:

Let $D(M)$ be the derived set of M.

Let $f \in D(M)$.

$\Rightarrow \exists\{f_n\} \in M$ converging to f

$\Rightarrow p(f_n - f) \to 0$ as $n \to \infty$

$\Rightarrow |(f_n - f)(k)| \to 0 \;\; \forall k$

$\Rightarrow |f_n(k) - f(k)| \to 0 \;\; \forall k$

$\Rightarrow f_n(k) \to f(k) \;\; \forall k$

$\Rightarrow \lim_{n\to\infty} f_n(k) = f(k)$.

Let $(k,l) = 1$. Then

$$f(kl) = \lim_{n\to\infty} f_n(kl) = \lim_{n\to\infty}\{f_n(k)\, f_n(l)\} = \lim_{n\to\infty} f_n(k) \cdot \lim_{n\to\infty} f_n(l) = f(k)\, f(l).$$

$\Rightarrow f$ is multiplicative.

$\Rightarrow f \in M$.

$\Rightarrow D(M) \subset M$

$\Rightarrow M$ is a closed subset of A.

Definition 2.3: (Cauchy Product)

$$(f \circ g)(n) = \sum_{k=0}^{n} f(k)g(n-k) \qquad ...(4)$$

In equation (4), n can take the value zero.

$$(f \circ g)(n) = f(0)g(n) + f(1)g(n-1) + ... + f(n)g(0).$$

Theorem 2.4:

$(M, 0)$ is an Abelian group with the identity $e(n) = \begin{cases} 1 & if \quad n = 0 \\ 0 & if \quad n > 0 \end{cases}$

Proof:

We shall verify $(f \circ e)(n) = f(0)e(n) + f(1)e(n-1) + ... + f(n)e(0)$

$$= 0 + 0 + ... + f(n) \cdot 1 = f(n).$$

$\Rightarrow$ $f \circ e = f$. Similarly $e \circ f = e$.

Hence e is the identity. Other axioms are verified similarly.

Definition 2.5: (Dirichlet Product)

$$(f \circ g)(n) = \sum_{d/n} f(d)g\left(\frac{n}{d}\right) \qquad ...(5)$$

Where the summation is overall divisors d of n .

Theorem 2.6:

$(M, \circ)$ is an Abelian group with the identity $e(n) = \begin{cases} 1 & if \quad n = 1 \\ 0 & if \quad n > 1 \end{cases}$

Definition 2.7:

Let $f(n) \neq 0 \;\; \forall n$, we call f a non-zero function.

Let $G = \{\text{all non-zero functions in M}\}$.

Define for $f, g \in G$, $(f * g)(n) = f(n) \cdot g(n)$.

Theorem 2.8:

$(G,*)$ is an Abelian group.

Proof:

Step 1:

Let $(m,n)=1$ and $m,n \in Z^+$.

$$\begin{aligned}(f*g)(mn) &= f(mn)\,g(mn)\\ &= f(m)\,f(n)\,g(m)\,g(n)\\ &= [f(m)\,g(m)][f(n)\,g(n)]\\ &= [(f*g)(m)][(f*g)(n)]\end{aligned}$$

$\Rightarrow (f*g) \in G$.

(i) Hence G is closed with respect to $*$.

Step 2:

Let $f,g,h \in G$.

$$\begin{aligned}[f*(g*h)](n) &= f(n)\cdot(g*h)(n)\\ &= f(n)[g(n)\cdot h(n)]\\ &= [f(n)\cdot g(n)]h(n)\\ &= [(f*g)(n)]h(n)\\ &= [(f*g)*h](n)\end{aligned}$$

$\Rightarrow f*(g*h)=(f*g)*h$.

(ii) Thus Associativity holds.

Step 3:

Define $e: Z^+ \to C$ by $e(n)=1\ \forall n$.

$e(mn)=1$

$= e(m) \cdot e(n)$ with $(m,n) = 1$.

$\Rightarrow\ e \in G$.

Also $(f * e)(n) = f(n) \cdot e(n) = f(n) \cdot 1 = f(n)$.

$\Rightarrow\ f * e = f$ and similarly $e \circ f = f$.

(iii) Hence e is the identity.

Step 4:

Given $f \in G$, take g as $g(n) = \dfrac{1}{f(n)}$.

$$g(mn) = \frac{1}{f(mn)} = \frac{1}{f(m)f(n)} = \frac{1}{f(m)} \cdot \frac{1}{f(n)} = g(m)g(n).$$

$\Rightarrow\ g \in G$.

Also $(f * g)(n) = f(n) \cdot g(n) = \dfrac{f(n)}{f(n)} = e \cdot n$.

$\Rightarrow\ f * g = e = g * f$.

(iv) Hence g is the inverse of f .

§3. Completely Multiplicative Functions

Definition 3.1:

f is completely multiplicative if

(1) $f(1) = 1$

(2) $f(mn) = f(m)f(n)$ if $\gcd(m,n) = 1$

(3) $f(p^k) = \{f(p)\}^k$ where p is a prime and k is positive integer.

Let $C = \{\text{all completely multiplicative functions}\}$.

Note: $C \subset M$.

Example:

Define $f(n) = n$.

$\Rightarrow f(p^k) = p^k = \{f(p)\}^k$

$\Rightarrow f$ is completely multiplicative.

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