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Regular Splitting Method for Approximating Linear System of Fuzzy Equation

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Abstract

A class of splitting iterative methods are considered for solving fuzzy system of linear equations, which covers Jacobi, Gauss Seidel, SOR, SSOR and their block variance proposed. Theoretical analysis showed that for a regular splitting, the corresponding iterative method converge to the unique fuzzy solution for any initial vector and fuzzy right hand side vector. Finally, we illustrate our approach by some numerical examples.

Keywords: Fuzzy number; Fuzzy linear system; Splitting iterative methods

1 Introduction

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [34], Dubois and Prade [11]. We refer the reader to [22] for more information on fuzzy numbers and fuzzy arithmetic. Fuzzy systems are used to study a variety of problems ranging from fuzzy topological spaces [9] to control chaotic systems [17, 20], fuzzy metric spaces [27], fuzzy differential equations [4], fuzzy linear systems [3, 2, 8, 31] and particle physics [12, 13, 14, 15, 16, 26, 29].

One of the major applications of fuzzy number arithmetic is treating fuzzy linear systems and fully fuzzy linear systems. Several problems in various areas such as economics, engineering and physics lead to the solution of a linear system of equations. In many applications, the parameters of the system (or at least some of them) should be represented by fuzzy rather than crisp numbers. Thus, it is very important to develop numerical procedures that can appropriately treat fuzzy linear systems and solve them.

Friedman *et al.* [18] introduced a general model for solving a fuzzy $n \times n$ linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. They used the parametric form of fuzzy numbers and replaced the original fuzzy $n \times n$ linear system by a crisp $2n \times 2n$ linear system and studied the duality in fuzzy linear systems $Ax = Bx + y$ where A and B are two real $n \times n$ matrices and the unknown x and the known y are two vectors whose components are n fuzzy numbers [19]. In [1, 3, 2, 8] the authors presented conjugate gradient and LU decomposition method for solving general fuzzy linear systems or symmetric fuzzy linear systems. Also, Wang *et al.* [33] presented an iterative algorithm for solving dual linear systems of the form $x = Ax + u$, where A is a real $n \times n$ matrix, the unknown x and the constant u are all vectors whose components are fuzzy numbers. Abbasbandy *et al.* [5] investigated the existence of a minimal solution of a general dual fuzzy linear system of the form $Ax + f = Bx + c$, where A and B are two real $m \times n$ matrices and the unknown x and the known f and c are vectors whose components are fuzzy numbers. Recently, Muzziloi *et al.* [25] considered fully fuzzy linear systems of the form $A_1x + b_1 = A_2x + b_2$ with A_1 and A_2 two square matrices of fuzzy entries and b_1 and b_2 fuzzy number vectors. Dehghan *et al.* [10] considered fully fuzzy linear systems of the form $Ax = b$ where A and b are a fuzzy matrix and a fuzzy vector, respectively and discussed the iterative solution of fully fuzzy linear systems.

In this paper, a class of splitting iterative methods are considered for solving fuzzy system of linear equations, which covers Jacobi, Gauss Seidel, SOR, SSOR and their block variance proposed. In Section 2, we recall some fundamental results on fuzzy numbers. In Section 3, the proposed model for solving the system $Ax = y$ where A is real $n \times n$ matrix, the unknown vector x is vector consisting of n fuzzy numbers and the constant y is vector consisting of n fuzzy numbers, are discussed. Numerical examples are given in Section 4 followed by a discussion and concluding in Section 5.

2 Preliminaries

In this section the basic notations used in fuzzy calculus are introduced. We start by defining the fuzzy number.

Definition 1. A fuzzy number is a fuzzy set $u : \mathbb{R}^1 \longrightarrow I = [0, 1]$ such that

i. u is upper semi-continuous.

ii. $u(x) = 0$ outside some interval $[a, d]$.

iii. There are real numbers b and c, $a \leq b \leq c \leq d$, for which

1. $u(x)$ is monotonically increasing on $[a, b]$,
2. $u(x)$ is monotonically decreasing on $[c, d]$,
3. $u(x) = 1, b \leq x \leq c$.

The set of all the fuzzy numbers (as given in definition 1) is denoted by E^1.

An alternative definition which yields the same E^1 is given by Kaleva and Ma [21, 24].

Definition 2. A fuzzy number u is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$ and $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfy the following requirements:

i. $\underline{u}(r)$ is a bounded monotonically increasing, left continuous function on $(0, 1]$ and right continuous at 0.

ii. $\overline{u}(r)$ is a bounded monotonically decreasing, left continuous function on $(0, 1]$ and right continuous at 0.

iii. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

A crisp number r is simply represented by $\underline{u}(\alpha) = \overline{u}(\alpha) = r, 0 \ \leq \alpha \leq 1$. The set of all the fuzzy numbers is denoted by E^1.

A popular fuzzy number is the triangular fuzzy number $u = (u_m, u_l, u_r)$ where u_m denotes the modal value and the real values $u_l > 0$ and $u_r > 0$ represent the left and right fuzziness, respectively. The membership function of a triangular fuzzy number is defined by:

$$\mu_u(x) = \begin{cases} \frac{x - u_m}{u_l} + 1, & u_m - u_l \leq x \leq u_m, \\ \frac{u_m - x}{u_r} + 1, & u_m \leq x \leq u_m + u_r, \\ 0, & otherwise. \end{cases}$$

Its parametric form is

$$\underline{u}(\alpha) = u_m + u_l(\alpha - 1), \quad \overline{u}(\alpha) = u_m + u_r(1 - \alpha).$$

Triangular fuzzy numbers are fuzzy numbers in LR representation where the reference functions L and R are linear.

3 The model

For arbitrary fuzzy numbers $x = (\underline{x}(r), \overline{x}(r)),\ y = (\underline{y}(r), \overline{y}(r))$ and real number k, we may define the addition and the scalar multiplication of fuzzy numbers by using the extension principle as [24]

(a) $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$,

(b) $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$,

(c) $kx = \begin{cases} (k\underline{x}, k\overline{x}), & k \geq 0, \\ (k\overline{x}, k\underline{x}), & k < 0. \end{cases}$

Definition 3. [18] A fuzzy number vector $(x_1, x_2, ..., x_n)^t$ given by

$$x_j = (\underline{x}_j(r), \overline{x}_j(r)); \quad j = 1, 2, \ldots, n, \ 0 \leq r \leq 1,$$

is called a solution of the fuzzy linear system $Ax = y$ if

$$\begin{cases} \underline{\sum_{j=1}^{n} a_{ij} x_j} = \sum_{j=1}^{n} \underline{a_{ij} x_j} = \underline{y}_i, \\ \overline{\sum_{j=1}^{n} a_{ij} x_j} = \sum_{j=1}^{n} \overline{a_{ij} x_j} = \overline{y}_i, \quad i = 1, 2, \ldots, n. \end{cases}$$

If, for a particular i, $a_{ij} > 0$, for all j, we simply get

$$\sum_{j=1}^{n} a_{ij} \underline{x}_j = \underline{y}_i, \qquad \sum_{j=1}^{n} a_{ij} \overline{x}_j = \overline{y}_i.$$

In general, however, an arbitrary equation for either $\underline{y}_i$ or $\overline{y}_i$ may include a linear combination of $\underline{x}_j$'s and $\overline{x}_j$'s. Consequently, in order to solve the system given $Ax = y$ one must solve a crisp $(2n) \times (2n)$ linear system where the right-hand side column is the function vector $(\underline{y}_1, \underline{y}_2, ..., \underline{y}_n, -\overline{y}_1, -\overline{y}_2, ..., -\overline{y}_n)^t$. We get the $(2n) \times (2n)$ linear system

$$\begin{cases} s_{11}\underline{x}_1 + \cdots + s_{1n}\underline{x}_n + \\ \qquad s_{1,n+1}(-\overline{x}_1) + \cdots + s_{1,2n}(-\overline{x}_n) & = \underline{y}_1, \\ \vdots \\ s_{n,1}\underline{x}_1 + \cdots + s_{nn}\underline{x}_n + \\ \qquad s_{n,n+1}(-\overline{x}_1) + \cdots + s_{n,2n}(-\overline{x}_n) & = \underline{y}_n, \\ s_{n+1,1}\underline{x}_1 + \cdots + s_{n+1,n}\underline{x}_n + \\ \qquad s_{n+1,n+1}(-\overline{x}_1) + \cdots + s_{n+1,2n}(-\overline{x}_n) & = -\overline{y}_1, \\ \vdots \\ s_{2n,1}\underline{x}_1 + \cdots + s_{2n,n}\underline{x}_n + \\ \qquad s_{2n,n+1}(-\overline{x}_1) + \cdots + s_{2n,2n}(-\overline{x}_n) & = -\overline{y}_n, \end{cases} \tag{1}$$

where s_{ij} are determined as follows:

$$\begin{aligned} a_{ij} \geq 0 &\Longrightarrow s_{ij} = a_{ij}, \ s_{i+n,j+n} = a_{ij}, \\ a_{ij} < 0 &\Longrightarrow s_{i,j+n} = -a_{ij}, \ s_{i+n,j} = -a_{ij}, \end{aligned} \tag{2}$$

and any s_{ij} which is not determined by (2) is zero. Using matrix notation we get

$$SX = Y, \tag{3}$$

where $S = (s_{ij}) \geq 0, 1 \leq i, j \leq 2n$ and

$$X = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\overline{x}_1 \\ \vdots \\ -\overline{x}_n \end{bmatrix}, \quad Y = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ -\overline{y}_1 \\ \vdots \\ -\overline{y}_n \end{bmatrix}.$$

The structure of S implies that $s_{ij} \geq 0, 1 \leq i, j \leq 2n$ and that

$$S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix},$$

where S_1 contains the positive entries of A, and S_2 contains the absolute values of the negative entries of A, i.e., $A = S_1 - S_2$.

4 Splitting iterative methods and its convergence

As we have known, a general splitting form of a $n \times n$ general matrix P could be expressed as follow:

$$S = M - N \tag{4}$$

where M and N are matrices.
If M is non-singular, we say that formula (4) represent a splitting of the matrix S, for solving the linear system

$$SX = Y \tag{5}$$

with this splitting is the iterative method

$$MX^{k+1} = NX^k + Y, \tag{6}$$

where k is number of iterations [30]. The above linear system could be written equivalently as

$$X^{k+1} = M^{-1}NX^k + M^{-1}Y. \tag{7}$$

Let $S_1 = D_1 - L_1 - U_1$ with D_1, L_1 and U_1 are diagonal, strictly lower and strictly upper triangular matrices respectively, then $S = D - L - U$, where

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_1 \end{pmatrix}, L = \begin{pmatrix} L_1 & 0 \\ S_2 & L_1 \end{pmatrix}, U = \begin{pmatrix} U_1 & S_2 \\ 0 & U_1 \end{pmatrix},$$

then M, N and the associated iterative method could be chosen as following:

$$M = \frac{1}{\alpha}D, \quad N = \frac{1}{\alpha}D - S,$$

with $\alpha > 0$, which gives Richardson's method.

$$M = D, \quad N = L + U,$$

which gives the Jacobi iterative method [6].

$$M = D - L, \quad N = U,$$

which gives the Gauss-Seidel iterative method [6].

$$M = \frac{1}{w}(D - wL), \quad N = \frac{1}{w}((1-w)D - wU),$$

which gives the successive over relaxation iterative methods [7].

$$M = \frac{1}{w(2-w)}(D - wL)D^{-1}(D - wU),$$

$$N = \frac{1}{w(2-w)}[(1-w)D + wL]D^{-1}[(1-w)D - wU],$$

which gives the symmetric successive over relaxation iterative methods [32]. Hermitian and skew-Hermitian splitting method is based on the splitting as follows

$$M = \frac{S + S^T}{2}, \quad N = \frac{S - S^T}{2}.$$

We should remark that the iterative scheme (6) converges for any initial vector, if and only if $\rho(M^{-1}N) < 1$ [30]. In particular when $a_{ii} > 0$ for $i = 1, 2, \ldots, n$ and matrix A is strictly diagonally dominant, then S is strictly diagonally dominant and the Jacobi and Gauss-Seidel iterative methods converge to the solution (5) for any initial value [6].
In the below, we consider a special splitting, called the regular splitting and analysis the convergence of the corresponding iterative methods for the fuzzy systems of linear equation.

Definition 4. For $2n \times 2n$ real matrices S, M and N, $S = M - N$ is a regular Splitting of the matrix S, if M is a nonsingular with $M^{-1} \geq 0$ and $N^{-1} \geq 0$. Similarly, $S = M - N$ is a weak regular splitting of the matrix S, if M is a nonsingular with $M^{-1} \geq 0$ and $M^{-1}N \geq 0$.

Here, a matrix $C \geq 0$ means that all the elements in C is a non-negative number, e.g., C is a non-negative matrix.
We should remark that a regular splitting of S is automatically a weak regular splitting of S, but the converse is not true in general.
Let $S = M - N$ be a regular splitting of the matrix S, the following theorem [30] gives the relationship between nonnegativity of S^{-1} and spectral radius of $M^{-1}N$.

Theorem 4.1. Let $S = M - N$ be a regular splitting of the matrix S. Then, S is nonsingular with S^{-1} if and only if $\rho(M^{-1}N) < 1$, where

$$\rho(M^{-1}N) = \frac{\rho(S^{-1}N)}{1 + \rho(S^{-1}N)}.$$

Thus, if S is nonsingular with $S^{-1} > 0$, the iterative method converges for any initial vector.
When $S = M - N$ be a weak regular splitting of the matrix S, we should remark that if $\rho(M^{-1}N) < 1$, then S is nonsingular and $S^{-1} > 0$, while the converse can not be true.
The following theorem gives the condition for the convergence of the corresponding iterative method for Eq.(7).

Theorem 4.2. Let $S = M - N$ be a regular splitting of the nonsingular matrix S with $S^{-1} \geq 0$, then the corresponding iterative method converges to the unique fuzzy solution for any initial vector and arbitrary fuzzy vector y.
Proof. See [18].

For a weak regular splitting $S = M - N$, we should remark that if $\rho(M^{-1}N) < 1$, the corresponding splitting iterative method converges for any fuzzy initial vector and in this case, we have S is nonsingular with S^{-1}, so that fuzzy unique solution is also a fuzzy vector for arbitrary fuzzy vector y. In practical, there is a number of choices of this splitting. Here we give two examples to further illustrate the above theorems. First, as for point Jacobi iterative method, let

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_1 \end{pmatrix}, L = \begin{pmatrix} L_1 + U_1 & S_2 \\ S_2 & L_1 + U_1 \end{pmatrix}, \tag{8}$$

where $S_1 = D_1 - L_1 - U_1 \geq 0$ and $S_2 \geq 0$. If S_1 is a diagonal matrix with positive diagonal, where $L_1 = U_1 = 0$, it is easy to see that $M^{-1} = diag\{D_1^{-1}, D_1^{-1}\} \geq 0$ and $N \geq 0$, thus this splitting is a regular splitting. Then, the following theorem gives the condition for the convergence of the splitting iterative method.

Theorem 4.3. For the nonsingular crisp matrix S with $S^{-1} > 0$,if S_1 is a diagonal matrix, and M and N are defined as (8), the corresponding iterative method converges to the unique fuzzy solution for any initial vector and arbitrary fuzzy vector y.

Consider the following block choice

$$M = \begin{pmatrix} S_1 & 0 \\ 0 & S_1 \end{pmatrix}, N_1 = \begin{pmatrix} 0 & S_2 \\ S_2 & 0 \end{pmatrix}, \tag{9}$$

with $N \geq 0$ due to $S_2 \geq 0$. Thus, this splitting is a regular splitting if and only if $M^{-1} \geq 0$, which follows that $S_1^{-1} \geq 0$. Since $S_1 \geq 0$ from its definition and $S_1^{-1} \geq 0$, from the theorem in [23]. $S_1 \geq 0$ should be a generalized permutation matrix.
Then, the theorem for the convergence of the splitting iterative method is given as follows.

Theorem 4.4. let M and N is 2×2 block matrices define above for nonsingular S with $S_1^{-1} \geq 0$ as well as $S_1 \geq 0$ be a generalized permutation matrix. Then, the corresponding iterative method is converges to the unique fuzzy solution for initial value and arbitrary fuzzy vector y.

5 Numerical examples

To illustrate the technique proposed in this paper, consider the following examples.

Example 5.1. Consider the following fuzzy linear system

$$\begin{cases} 2x_1 - x_2 = (3 + r, 5 - r), \\ 3x_2 = (2 + 3r, 6 - r). \end{cases}$$

The extended 4×4 matrix is

$$S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}$$

where

$$S_1 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we chose

$$M = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix} \geq 0, \quad N = \begin{pmatrix} N_1 & N_2 \\ N_2 & N_1 \end{pmatrix} \geq 0,$$

where

$$M_1 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$N_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

where $M^{-1} \geq 0$.
Thus, $S = M - N'$, where $N' = -N$ is a regular splitting of a crisp matrix S. In addition, $\rho(M^{-1}N) = 0$, so this splitting iterative method can converge to unique solution

$$x_1 = (2.5 + 0.333r, 2.8333),$$
$$x_2 = (0.6667 + r, 2 - 0.333r).$$

Here $\underline{x}_1 \leq \overline{x}_1, \underline{x}_2 \leq \overline{x}_2$; $\underline{x}_1, \underline{x}_2$ are monotonic increasing functions and $\overline{x}_1, \overline{x}_2$ are monotonic decreasing functions. Therefore the fuzzy solution is $x_1 = (\underline{x}_1, \overline{x}_1), x_2 = (\underline{x}_2, \overline{x}_2)$ is a strong fuzzy solution.

Example 5.2. Consider the following fuzzy linear system

$$\begin{cases} 8x_1 + 2x_2 + x_3 - 3x_5 = (r, 2 - r), \\ -2x_1 + 5x_2 + x_3 - x_4 + x_5 = (4 + r, 7 - 2r), \\ x_1 - x_2 + 5x_3 + x_4 + x_5 = (1 + 2r, 6 - 3r), \\ -x_3 + 4x_4 + 2x_5 = (1 + r, 3 - r), \\ x_1 - 2x_2 + 3x_5 = (3r, 6 - 3r). \end{cases}$$

The extended 10×10 matrix is

$$S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}$$

where

$$S_1 = \begin{pmatrix} 8 & 2 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 & 1 \\ 1 & 0 & 5 & 1 & 1 \\ 0 & 0 & 0 & 4 & 2 \\ 1 & 0 & 0 & 0 & 3 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{pmatrix},$$

we chose

$$M = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix} \geq 0, \quad N = \begin{pmatrix} N_1 & N_2 \\ N_2 & N_1 \end{pmatrix} \geq 0,$$

where $M_1 = S_1$, $M_2 = 0$, and $N_1 = 0$, $N_2 = S_2$.
Thus, $S = M - N'$, where $N' = -N$ is a regular splitting of a crisp matrix S. In addition, $\rho(M^{-1}N) = 0.56188$, so this splitting iterative method with 10 step, can converge to the

$$\begin{aligned}
x_1 &= (0.72882 - 0.33061r, 0.044174 + 0.3539r),\\
x_2 &= (0.6143 + 0.16628r, 1.0775 - 0.29689r),\\
x_3 &= (0.12645 + 0.2906r, 0.91852 - 0.50144r),\\
x_4 &= (0.2419 - 0.3314r, -0.41654 + 0.32697r),\\
x_5 &= (0.47497 + 0.91243r, 2.3948 - 1.007r),
\end{aligned}$$

and the solution is a weak solution.

6 Summary and conclusions

In this work, we analysis a class of splitting iterative method for the solution of fuzzy system of linear equations, which covers Jacobi, Gauss Seide, SOR, SSOR and their block variance proposed. Theoretical analysis showed that for a regular splitting, the corresponding iterative method converge the unique fuzzy solution for any initial vector and fuzzy right hand side vector.

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