Int. J. Contemp. Math. Sciences, Vol. 5, 2010, no. 6, 253 - 261

On Fully Pseudo (m, n)-Stable Modules

Muna J. MohammedAli

Department of Mathematics, College of Science for Women
University of Baghdad, Baghdad, Iraq
a73n79a80@yahoo.com

Abstract

Let R be a commutative ring with non-zero identity element. For two fixed positive integers m and n. A right R-module M is called fully pseudo (m, n)-stable, if $\theta(N) \subseteq N$ for each n-generated submodule N of M^m and R-monomorphism $\theta : N \to M^m$. In this paper we give some characterization theorems and properties of fully pseudo (m, n)-stable modules which generalize the results of fully pseudo stable modules.

Mathematics Subject Classification(2000): 13C99, 13C11

Keywords: fully (m, n)-stable module, fully pseudo (m, n)-stable module, (m, n)-multiplication module.

1 Introduction

Throughout, R is a commutative ring with non-zero identity and all modules are unitary. We use the notation $R^{m\times n}$ for the set of all $m \times n$ matrices over R. For $A \in R^{m\times n}$,A^T will denote the transpose of A. In general, for an R-module N, we write $N^{m\times n}$ for the set of all formal $m\times n$ matrices whose entries are elements of N. Let M be a right R-module and N be a left R-module. For $x \in M^{l\times m}$, $s \in R^{m\times n}$ and $y \in N^{n\times k}$, under the usual multiplication of matrices, xs (resp. sy) is a well defined element in $M^{l\times m}$ (resp. $N^{n\times k}$). If $X \in M^{l\times m}$, $S \in R^{m\times n}$ and $Y \in N^{n\times k}$, define

$$\ell_{M^{l\times m}}(S) = \{u \in M^{l\times m} : us = 0, \forall s \in S\}$$

$$r_{N^{n\times k}}(S) = \{v \in N^{n\times k} : sv = 0, \forall s \in S\}$$

$$\ell_{R^{m\times n}}(Y) = \{s \in R^{m\times n} : sy = 0, \forall y \in Y\}$$

$$r_{R^{m\times n}}(X) = \{s \in R^{m\times n} : xs = 0, \forall x \in X\}$$

We will write $N^n = N^{1\times n}$, $N_n = N^{n\times 1}$. Fully pseudo stable module have been discussed in [1], an R-module M is called fully pseudo stable, if $\theta(N) \subseteq N$ for each submodule N of M and R-monomorphism θ from N into M. It is an easy matter to see that M is fully pseudo stable, if and only if $\theta(xR) \subseteq xR$ for each x in M and R-monomorphism $\theta : xR \to M$. In this paper, for two fixed positive integers m and n, we introduce the concepts of fully pseudo (m, n)-stable modules, if and only if the distinct n-generated submodules of M^m are not isomorphic.

2 Results

Definition 2.1 *An R-module M is called fully pseudo (m, n)-stable, if $\theta(N) \subseteq N$ for each n-generated submodule N of M^m and R-monomorphism $\theta : N \to M^m$. The ring R is fully pseudo (m, n)-stable, if R is fully pseudo (m, n)-stable as R-module.*

An R-module M is fully pseudo (m, n)-stable, if and only if for each R-monomorphism $\theta : N(= \sum_{i=1}^{n} \alpha_i R) \to M^m$ (where $\alpha_i \in M^m$) and each $w \in N$, there exists $t = (t_1, \ldots, t_n) \in R^n$ such that $\theta(w) = \sum_{i=1}^{n} \alpha_i t_i = (\alpha_1, \ldots, \alpha_n)t^T$, if $r = (r_1, \ldots, r_n) \in R^n$, then $\theta((\alpha_1, \ldots, \alpha_n)r^T) = (\alpha_1, \ldots, \alpha_n)t^T$.

It is clear that M is fully pseudo $(1, 1)$-stable, if and only if M is fully pseudo stable.

It is an easy matter to see that an R-module M is fully pseudo (m, n)-stable, if and only if it is fully pseudo (m, q)-stable for all $1 \le q \le n$, if and only if it is fully pseudo (p, n)-stable for all $1 \le p \le m$, if and only if it is fully pseudo (p, q)-stable for all $1 \le p \le m$ and $1 \le q \le n$.

Recall that an R-module M is called fully (m, n)-stable, if $\theta(N) \subseteq N$ for each n-generated submodule N of M^m and R-homomorphism $\theta : N \to M^m$. The ring R is fully (m, n)-stable, if R is fully (m, n)-stable as R-module.It is clear that every fully (m, n)-stable is fully pseudo (m, n)-stable. But the converse is not true.

Recall that an R-module M is uniform if any non-zero submodules of M has non-zero intersection.

Proposition 2.2 *Every uniform fully pseudo (m, n)-stable R-module is fully (m, n)-stable*

proof. let M be fully pseudo (m, n)-stable module. For any n-generated submodule N of M^m and R-homomorphism $\theta : N \to M^m$. If $ker\theta = 0$, nothing to prove. Otherwise let $x \in ker\theta \cap ker(I_N + \theta)$, then $\theta(x) = 0$ and $(I_N + \theta)(x) = 0$. Now $x = x + \theta(x) = (I_N + \theta)(x) = 0$. Thus $ker\theta \cap ker(I_N + \theta) = 0$. But M is uniform, hence $ker(I_N + \theta) = 0$, that is $(I_N + \theta) : N \to M^m$ is an R-monomorphism. Since M is fully pseudo (m, n)-stable, then $(I_N + \theta)(N) \subseteq (N)$, hence $\theta(N) \subseteq (N)$.

Corollary 2.3 *[1, Proposition 2.2]Every uniform fully pseudo stable R-module is fully stable module.*

Theorem 2.4 *An R-module M is fully pseudo (m, n)-stable, if and only if distinct n-generated submodules of M^m are not isomorphic.*

proof. suppose that distinct n-generated submodules of M^m are not isomorphic, and there exists an n-generated submodule N of M^m and R-monomorphism $\theta : N \to M^m$ such that $\theta(N) \not\subseteq N$, then N and $\theta(N)$ are two distinct n-generated submodules of M^m. By assumption, then $\theta(N)$ is not isomorphic to N which is an absurd. Conversely, suppose that M is a fully pseudo (m, n)-stable and M has two n-generated submodules N_1 and N_2 such that $N_1 \cong N_2$. No loss of generality if it is assumed that $N_1 \not\subseteq N_2$. There exists a non-zero element x in N_1 not in N_2. Let $\theta : N_1 \to N_2$ be an isomorphism ,consider the following two R-monomorphism $i_{N_1} \circ \theta : N_1 \to M^m$ and $i_{N_1} \circ \theta^{-1} : N_2 \to M^m$. Since M is fully pseudo(m, n)-stable, then $(i_{N_2} \circ \theta)(N_1) \subseteq N_1$ and $(i_{N_1} \circ \theta^{-1})(N_2) \subseteq N_2$. Now $x = (i_{N_1} \circ \theta^{-1} \circ i_{N_2} \circ \theta)(x) \in N_2$ which is contradiction .

Corollary 2.5 *[1, Proposition 2.4]An R-module M is fully pseudo-stable, if and only if distinct cyclic submodules of M are not isomorphic.*

Corollary 2.6 *Let M be a uniform R-module. Then M is fully (m, n)-stable, if and only if distinct n-generated submodules of M^m are not isomorphic.*

In [7], prove that, let M be a right R-module and I_R an n-generated submodule of R_R^m, then $\ell_{M_m}(I) \cong Hom_R(R^m/I, M)$.

Theorem 2.7 *Let M be an R-module. Then the following statement are equivalent.*

1. *M is fully (m,n)-stable.*

2. *distinct n-generated submodules of M^m are not isomorphic and $\sum_{i=1}^{n}\alpha_i R \cong Hom_R(\sum_{i=1}^{n}\alpha_i R, M)$ for each n-elements subset $\{\alpha_1,\ldots,\alpha_n\} of M^m$*

proof. Assume that M is fully (m,n)-stable R-module, then the distinct n-generated submodules of M^m are not isomorphic. By [2, proposition (2.9)],for each n-element subset $\{\alpha_1,\ldots,\alpha_n\} \in M^m$, we have $\alpha_1 R + \ldots + \alpha_n R = \ell_{M^m} r_{R_m}(\alpha_1 R + \ldots + \alpha_n R) \cong Hom_R(R^m / r_{R_m}(\alpha_1 R + \ldots + \alpha_n R), M)$

$$\cong Hom_R((\alpha_1 R + \ldots + \alpha_n R), M).$$

Conversely, assume that distinct n-generated submodules of M^m are not isomorphic and $\alpha_1 R + \ldots + \alpha_n R \cong Hom_R((\alpha_1 R + \ldots + \alpha_n R), M)$. By [6], we have $\ell_{M^m}(r_{R_m}(\alpha_1 R + \ldots + \alpha_n R)) \cong Hom_R((\alpha_1 R + \ldots + \alpha_n R), M)$, then $\alpha_1 R + \ldots + \alpha_n R \cong \ell_{M^m} r_{R_m}(\alpha_1 R + \ldots + \alpha_n R)$. distinct n-generated submodules of M^m are not isomorphic implies that $\alpha_1 R + \ldots + \alpha_n R = \ell_{M^m} r_{R_m}(\alpha_1 R + \ldots + \alpha_n R)$. Hence M is fully (m,n)-stable.

Corollary 2.8 *[1, Theorem 2.8] Let M be an R-module. Then the following statement are equivalent.*

1. *M is fully-stable.*

2. *distinct cyclic submodules of M are not isomorphic and $xR \cong Hom_R(xR, M)$ for each $x \in M$*

Corollary 2.9 *Let M be an R-module. Then the following statement are equivalent.*

1. *M is a fully (m,n)-stable.*

2. *M is a fully pseudo (m,n)-stable and $\sum_{i=1}^{n}\alpha_i R \cong Hom_R(\sum_{i=1}^{n}\alpha_i R, M)$ for each n-elements subset $\{\alpha_1,\ldots,\alpha_n\} of M^m$*

Proposition 2.10 *Let M be an R-module. Then the following statement are equivalent.*

1. *distinct n-generated submodules of M^m are not isomorphic.*

2. *$r_{R_n}\{\alpha_1,\ldots,\alpha_n\} = r_{R_n}\{\beta_1,\ldots,\beta_n\}$ for each n-element two subsets $\{\alpha_1, \ldots,\alpha_n\}$ and $\{\beta_1,\ldots,\beta_n\}$ of M^n implies that $\alpha_1 R + \ldots + \alpha_n R = \beta_1 R + \ldots + \beta_n R$*

proof. Assume that (1) is true. Define $\theta : \alpha_1 R + \cdots + \alpha_n R \to \beta_1 R + \cdots + \beta_n R$ by $\theta(\sum_{i=1}^{n} \alpha_i r_i) = \sum_{i=1}^{n} \beta_i r_i$ for each $r_i \in R$ and $i = 1, \cdots, n$. Because $r_{R_n}\{\alpha_1, \ldots, \alpha_n\} = r_{R_n}\{\beta_1, \cdots, \beta_n\}$, then θ is an isomorphism, then by (1), we have $\alpha_1 R + \ldots + \alpha_n R = \beta_1 R + \ldots + \beta_n R$. Conversely, assume that (2) holds, and there exists two n-generated submodules $\alpha_1 R + \ldots + \alpha_n R \neq \beta_1 R + \ldots + \beta_n R$ with $\alpha_1 R + \ldots + \alpha_n R \cong \beta_1 R + \ldots + \beta_n R$, then without lose of generality there exists an element $w \in \alpha_1 R + \ldots + \alpha_n R$ and $w \notin \beta_1 R + \cdots + \beta_n R$. Let $f : \alpha_1 R + \cdots + \alpha_n R \to \beta_1 R + \cdots + \beta_n R$ be isomorphism. Now $w \neq f(w)$, otherwise $w \in \beta_1 R + \cdots + \beta_n R$. We claim that $r_{R_n}(w) = r_{R_n}(f(w))$. For let $\eta \in r_{R_n}(w)$, then $w\eta = 0$, hence $f(w)\eta = 0$, thus $r_{R_n}(w) \subseteq r_{R_n}(f(w))$. Let $\zeta \in r_{R_n}(f(w))$, then $0 = f(w)\zeta = f(w\zeta)$, $w\zeta \in ker(f) = 0$, $w\zeta = 0$ or $\zeta \in r_{R_n}(w)$. Therefore $r_{R_n}(w) = r_{R_n}(f(w))$. But $Rw \neq Rf(w)$ which is a contradiction.

Corollary 2.11 *[1, Proposition 2.11]Let M be an R-module . Then the following statements are equivalent.*

1. *distinct cyclic submodules are not isomorphic.*
2. *$r_R(x) = r_R(y)$ implies $Rx = Ry$ for each $x, y \in M$*

S. K. Jain and S. Singh in[4] introduced the concept of a pseudo-injective module . An R-module M is said to be pseudo-injective, if each R-monomorphism $\theta : N \to M$ of any submodule N of M can be extended to an R-endomorphism of M. An R-module M is said to be principally pseudo-injective, if each R-monomorphism from cyclic submodule N of M can be extended to an R-endomorphism of M [4].

Lemma 2.12 *Every fully pseudo-stable module is principally pseudo injective module.*

proof. Is clear

Motivated by concept of principally pseudo-injective, we introduce the following definition.

Definition 2.13 *An R-module M is called (m,n)-pseudo injective, if each R-monomorphism from n-generated submodule of M^m to M can be extended to an R-homomorphism from M^m to M.*
It is clear that M is principally pseudo-injective, if and only if M is (1,1)-pseudo injective. An R-module M is called n-pseudo injective if it is $(1,n)$-pseudo injective for all positive integers n.

It is an easy matter to see that an R-module M is (m, n)-pseudo injective , if and only if it is (m, q)-pseudo injective for all $1 \leq q \leq n$, if and only if it is (p, n)-pseudo injective for all $1 \leq p \leq m$, if and only if it is (p, q)-pseudo injective for all $1 \leq p \leq m$ and $1 \leq q \leq n$.

In [9] prove the following proposition.Write $A_m = \{n \in M \mid r_R(n) = r_R(m)\}$ and $B_m = \{\alpha \in S \mid ker\alpha \cap mR = 0\}$ for each $m \in M$.

Proposition 2.14 *Let M be an R-module. The following statements are equivalent for each $m \in M$.*

1. *M is principally pseudo injective*
2. *$A_m = B_m m$*
3. *If $A_m = A_n$ then $B_m m = B_n n$.*
4. *For every R-monomorphism $\alpha : 0 \to mR \to M$ and $\beta : 0 \to mR \to M$, there exists $\gamma \in End(M_R)$ such that $\alpha = \gamma\beta$.*

Let M be an R-module, α_i be a non-zero element in M^n, $i = 1, \ldots, m$. We write $A_{\alpha_i} = \{\beta_i \in M^n \mid r_{R_n}(\beta_i) = r_{R_n}(\alpha_i)\}$ and $B_{\alpha_i} = \{c \in S_n \mid r_{R^n}(c) \cap \alpha_i R = 0\}$.

Theorem 2.15 *Let M be an R-module. The following statements are equivalent*

1. *M is (m,n)-pseudo injective.*
2. *$A_{\alpha_i} = B_{\alpha_i}\alpha_i$, for each $\alpha_i \in M^n$.*

2′ *$A_U = B_U U$, for each $U \in M^{m\times n}$.*

3. *If $A_{\alpha_i} = A_{\beta_i}$ then $B_{\alpha_i}\alpha_i = B_{\beta_i}\beta_i$.*

3′ *If $A_U = A_V$ then $B_U U = B_V V$ for each $U, V \in M^{m\times n}$.*

4. *For every R-monomorphism $\theta : \sum_{i=1}^{n} \alpha_i R \to M^m$ and $\varphi : \sum_{i=1}^{n} \alpha_i R \to M$, there exists $\gamma : M^m \to M$ such that $\theta = \gamma\varphi$.*

proof. (1)⇒(2) $\theta : \alpha_1 R + \ldots \alpha_n R \to M$ is well-defined by $\theta(\sum_{i=1}^{n} \alpha_i r_i) = \sum_{i=1}^{n} \beta_i r_i$. for each $r_i \in R$. (m,n)- pseudo-injectivity of M implies there exists $\gamma : M^m \to M$ such that $\theta = \gamma\ i$. In particular, there is $c = (c_1, \ldots, c_n) \in S_n$ with $\beta_i = \sum_{k=1}^{n} c_k \alpha_i$, $i = 1, \ldots, n$. If $\sum_{i=1}^{n} \alpha_i r_i \in r_{R^n}(c) \cap \alpha_i R$, then

$$\gamma(\sum_{i=1}^{n} \alpha_i r_i) = \theta(\sum_{i=1}^{n} \alpha_i r_i) = \sum_{i=1}^{n} \beta_i r_i = \sum_{i=1}^{n}(\sum_{k=1}^{n} c_k \alpha_i) r_i$$

$$= \sum_{k=1}^{n} c_k(\sum_{i=1}^{n} \alpha_i r_i) = 0$$

so $\sum_{i=1}^{n} \alpha_i r_i = 0$, thus $c \in B_{\alpha_i}$ and hence $\beta_i \in B_{\alpha_i}\alpha_i$. Conversely if $\beta_i \in B_{\alpha_i}\alpha_i$, then $\beta_i = \sum_{k=1}^{n} s_k\alpha_i$ where $s = (s_1, \dots, s_n) \in S_n$ and $t \in r_{R^n}(s) \cap \alpha_i R = 0$. It is clear that $r_{R_n}(\{\alpha_1, \dots, \alpha_n\}) \subseteq r_{R_n}(\{s\alpha_1, \dots, s\alpha_n\})$.
If $t \in r_{R_n}(\{s_1\alpha_1, \dots, s_n\alpha_n\})$, then $\sum_{k=1}^{n} t_k(\sum_{i=1}^{n} s_i\alpha_i)$ and hence $\sum_{k=1}^{n} t_k\alpha_i \in r_{R^n}(s) \cap \alpha_i R$, so $\sum_{i=1}^{n} t_k\alpha_i = 0$. Thus $r_{R_n}(\{\alpha_1, \dots, \alpha_n\}) = r_{R_n}(\{\beta_1, \dots, \beta_n\})$. The other equivalence in (2) follows by symmetry.

(2)⇔(2′)and (3)⇔(3′) are trivial.

(2) ⇒(3) Let $A_{\alpha_i} = A_{\beta_i}$. Then $A_{\alpha_i} = B_{\alpha_i}\alpha_i$, $A_{\beta_i} = B_{\beta_i}\beta_i$. So $B_{\alpha_i}\alpha_i = B_{\beta_i}\beta_i$.

(3)⇒(4) For each $\alpha_i \in M^n, i = 1, \dots, n$, let $\theta : \sum_{i=1}^{n} \alpha_i R \to M^m$ and $\varphi : \sum_{i=1}^{n} \alpha_i R \to M$ be R-monomorphisms. Then by (3), $r_{R_n}(\varphi\alpha_i) = r_{R_n}(\theta\alpha_i)$. So $A_{\theta\alpha_i} = A_{\varphi\alpha_i}$, $B_{\theta\alpha_i}\theta\alpha_i = B_{\varphi\alpha_i}\varphi\alpha_i$. Because $r_{R_n}(1_{M^n}) \cap \alpha_i R = 0$, $1_{M^n} \in B_{\theta\alpha_i}$. Then $\theta\alpha_i \in B_{\varphi\alpha_i}\varphi\alpha_i$. There exists $\gamma \in B_{\varphi\alpha_i}$ such that $\theta = \gamma\varphi$

(4)⇒(1)Let $\varphi = i_{\sum_{i=1}^{n} \alpha_i R}$. It is clear.

Theorem 2.16 *Given an R-module M_R . Then M_R is (m,n)-pseudo injective, if and only if the right $R^{n\times n}$ -module $M^{m\times n}$ is principally pseudo-injective.*

proof. ⇒ Let $U, V \in M^{m\times n}$ with $r_{R^{n\times n}}(U) = r_{R^{n\times n}}(V)$ and write $V = \begin{pmatrix} V_1 \\ \vdots \\ V_m \end{pmatrix}$. Then for each $i = 1, \dots, m$, $r_{R^{n\times n}}(U) = r_{R^{n\times n}}(V_i)$. consequently, $r_{R_n}(U) = r_{R_n}(V_i)$. Since M is (m,n)-pseudo injective, by theorem (2.15), $B_U U = B_{V_i}V_i$, put $B_V V = \begin{pmatrix} B_{V_1}V_1 \\ \vdots \\ B_{V_m}V_m \end{pmatrix}$.So $A_U = B_V V$. Therefore the right $R^{n\times n}$ -module $M^{m\times n}$ is principally pseudo-injective by [9, proposition (2.1)].

⇐ Suppose that $\alpha_i, \beta_i \in M^n$ and $r_{R_n}(\alpha_i) = r_{R_n}(\beta_i)$. Let $U = \begin{pmatrix} \alpha_i \\ 0 \end{pmatrix}$ and $V = \begin{pmatrix} \beta_i \\ 0 \end{pmatrix} \in M^{m\times n}$. Then $r_{R^{n\times n}}(V) = r_{R^{n\times n}}(U)$. Since $M^{m\times n}_{R^{n\times n}}$ is principally pseudo-injective , $A_V V = A_U U$. Then M is (m,n)-pseudo-injective by theorem(2.15).

Let M be an R-module, α_i be a non-zero element in M^n, $i = 1, \ldots, m$ and t in R^n. We write $W(\alpha_i) = \{r \in R_n \mid \ell_{M^n}(r) \cap \alpha_i R = 0\}$.

Theorem 2.17 *The following are equivalent for an R-module M*

1. *M is fully pseudo (m,n)-stable.*
2. $r_{R_n}(\{\alpha_1, \ldots, \alpha_n\}) = r_{R_n}(\{\beta_1, \ldots, \beta_n\})$ *if and only if* $\beta_i \in \alpha_i W(\alpha_i)$ *if and only if* $\alpha_i \in \beta_i W(\beta_i)$ *for each two n-element subsets* $\{\alpha_1, \ldots, \alpha_n\}$ *and* $\{\beta_1, \ldots, \beta_n\}$ *of* M^n.

2′ $r_{R_n}(A) = r_{R_n}(B)$ *if and only if* $B \in AW(A)$ *if and only if* $A \in BW(B)$ *for each* $A, B \in M^{m\times n}$

3. *For any R-monomorphisms* $\theta, \varphi : \alpha_1 R + \ldots \alpha_n R \to M^m$ *where* $\alpha_i \in M^n$, *there is* $t \in R^n$ *such that* $\theta = \varphi \cdot t$.

proof. (1)⇒(2) $\theta : \alpha_1 R + \ldots + \alpha_n R \to M^m$ is well-defined by $\theta(\sum_{i=1}^n \alpha_i r_i) = \sum_{i=1}^n \beta_i r_i$. for each $r_i \in R$. Full pseudo (m,n)- stability of M implies $\theta(\sum_{i=1}^n \alpha_i R) \subseteq \sum_{i=1}^n \alpha_i R$. In particular, there is $t = (t_1, \ldots, t_n) \in R_n$ with $\beta_i = \sum_{k=1}^n \alpha_i t_k$, $i = 1, \ldots, n$. If $\sum_{i=1}^n \alpha_i r_i \in \ell_{M^n}(t) \cap \alpha_i R$, then $0 = \sum_{k=1}^n (\sum_{i=1}^n \alpha_i r_i)\ t_k = \sum_{i=1}^n (\sum_{k=1}^n \alpha_i t_k)\ r_i = \sum_{i=1}^n \beta_i r_i = \theta(\sum_{i=1}^n \alpha_i\ r_i)$, so $\sum_{i=1}^n \alpha_i r_i = 0$, thus $t \in W(\alpha_i)$ and hence $\beta_i \in \alpha_i W(\alpha_i)$. Conversely if $\beta_i \in \alpha_i W(\alpha_i)$, then $\beta_i = \sum_{k=1}^n \alpha_i s_k$ where $s = (s_1, \ldots, s_n) \in R_n$ and $\ell_{M^n}(s) \cap \alpha_i R = 0$. It is clear that $r_{R_n}(\{\alpha_1, \ldots, \alpha_n\}) \subseteq r_{R_n}(\{\alpha_1 s, \ldots, \alpha_n s\})$. If $t \in r_{R_n}(\{\alpha_1 s_1, \ldots, \alpha_n s_n\})$, then $\sum_{k=1}^n (\sum_{i=1}^n \alpha_i s_i) t_k$ and hence $\sum_{k=1}^n \alpha_i t_k \in \ell_{M^n}(s) \cap \alpha_i R$, so $\sum_{i=1}^n \alpha_i t_k = 0$. Thus $r_{R_n}(\{\alpha_1, \ldots, \alpha_n\}) = r_{R_n}(\{\beta_1, \ldots, \beta_n\})$. The other equivalence in (2) follows by symmetry.

(2) ⇔ (2′) Is trivial

(2) ⇒(3) Let $\theta, \varphi : \alpha_1 R + \ldots + \alpha_n R \to M^m$ be two R-monomorphisms. Then

$$r_{R_n}(\{\alpha_1, \ldots, \alpha_n\}) = r_{R_n}(\{\beta_1, \ldots, \beta_n\}).$$

By the hypothesis, $\theta(\alpha_i) \in \varphi(\alpha_i) W(\varphi(\alpha_i))$. So there is $t \in W(\varphi(\alpha_i))$, such that $\theta(\alpha_i) = \sum_{i=1}^n \varphi(\alpha_i) t_i = \sum_{i=1}^n \varphi(\alpha_i t_i)$. Also by symmetry there is $s \in R_n$ such that $\varphi = \theta s$.

(3)⇒(1) For each $\alpha_i \in M^n, i = 1, \ldots, m$, let $f : \alpha_1 R + \ldots \alpha_n R \to M^m$ be an R-monomorphism. Then by (3), there is an element $t \in R^n$ such that $f = it$ where i is the inclusion map of $\alpha_1 R + \ldots \alpha_n R$ into M^n and hence $f(\alpha_1 R + \ldots \alpha_n R) \subseteq \alpha_1 R + \ldots + \alpha_n R$.

Theorem 2.18 *Given an R-module M_R . Then M_R is fully pseudo (m,n)-stable, if and only if the right $R^{n\times n}$ -module $M^{m\times n}$ is fully pseudo-stable.*

proof. The proof is similar to the proof of theorem(2.16)

Theorem 2.19 *Given an R-module M_R. If M_R is fully pseudo-(m,n)-stable, then M is (m,n)-pseudo injective.*

The following proposition is the converse of theorem (2.19)

Proposition 2.20 *Let M be an (m,n)-multiplication R-module. If M is (m,n)-pseudo injective, then M is a fully pseudo (m,n)-stable module.*

proof. The proof is essentially the same as that of [2, proposition (2.17)] by replacing the R-homomorphism $f : N \to M^m$ by R-monomorphism.

References

[1] Abbas, M. S., *On fully stable modules,* Ph.D. Thesis, College of Science, University of Baghdad 1991.

[2] Abbas, M. S.and MohammedAli, M. J.,*A note on fully(m,n)-stable modules, International Electronic Journal of Algebra, vol* **6**,(2009), 65-73.

[3] chen,J.,Ding,N., Li,Y. and Zhou, Y., *On (m,n)- injectivity of Modules, Comm. in Algwbra,* **29**(12), (2001), 5589-5603.

[4] Jain, S. K and Singh, S.,*Quasi-injective and pseudo-injectve modules,Canad. Math. Bull. Vol.* **18** (3),(1975),359-365.

[5] Mohammedali, M. J.,*Some classes of injectivity and projectivity,*Ph.D. Thesis, College of Science, University of Baghdad 2006.

[6] Nicholson W. K., Park J. K. and Yousif M. F.,*Principally quasi-injective modules, Comm. Algebra.* **27**(4),(1999),1683-1693.

[7] Zhang, X., Chen J. and Zhang J., *On (m,n) injective modules and (m,n)-coherent rings, Algebra Colloquium* **12**(1), (2005),149-160.

[8] Zhu, Z. M., Chen, L. and Zhang, X. X.,*On (m,n)-quasi- injective modules, Acta Math. Univ. comenianae,* **1**, (2005), 25-36.

[9] X. Du and C. Zhoo, *Pseudo-injective modules and principally pseudo-injective modules,J. Math. research and exposition, vol.***27**,No.2,(2007),223-228.

Received: March, 2009