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Join Preserving Maps and Various Concepts

Yong Chan Kim

Department of Mathematics, Gangneung-Wonju University
Gangneung, Gangwondo 210-702, Korea
yck@gwnu.ac.kr

Jin Won Park

Department of Mathematics Educations, Cheju National University
Jeju, Jejudo, 690-756, Korea
jinwon@cheju.ac.kr

Abstract

In this paper, we investigate the properties of isotone (antitone) Galois connection. Moreover, we show that join preserving maps induce formal, attribute oriented and object oriented concepts on a complete residuated lattice.

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1 Introduction

Formal concept analysis is an important mathematical tool for data analysis and knowledge processing [1-4,7,9]. A formal concept consists of (X, Y, R) where X is a set of objects, Y is a set of attributes and R is a relation between X and Y. Bělohlávek [1-4] developed the notion of formal concepts with $R \in L^{X \times Y}$ on a complete residuated lattice L.

In this paper, we investigate the properties of isotone (antitone) Galois connections. Using their properties, we define formal, attribute oriented and object oriented concepts on a complete residuated lattice. Moreover, we show that join preserving maps induce formal, attribute oriented and object oriented concepts on a complete residuated lattice.

2 Preliminaries

Definition 2.1 [4,8,9] A triple $(L, \leq, \odot)$ is called a *complete residuated lattice* iff it satisfies the following conditions:

(L1) $L = (L, \leq, 1, 0)$ is a complete lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(L2) $(L, \odot, 1)$ is a commutative monoid;

(L3) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i \in \Gamma} a_i) \odot b = \bigvee_{i \in \Gamma} (a_i \odot b).$$

Define an operation $\rightarrow$ as $a \rightarrow b = \bigvee \{c \in L \mid a \odot c \leq b\}$, for each $a, b \in L$.

Example 2.2 [4,8,9] (1) Each frame $(L, \leq, \wedge)$ is a complete residuated lattice.

(2) The unit interval with a left-continuous t-norm t, $([0,1], \leq, t)$, is a complete residuated lattice.

(3) Define a binary operation $\odot$ on $[0,1]$ by $x \odot y = \max\{0, x + y - 1\}$. Then $([0,1], \leq, \odot)$ is a complete residuated lattice.

Let $(L, \leq, \odot)$ be a complete residuated lattice. A order reversing map $^* : L \rightarrow L$ defined by $a^* = a \rightarrow 0$ is called a *strong negation* if $a^{**} = a$ for each $a \in L$.

In this paper, we assume $(L, \leq, \odot, ^*)$ is a complete residuated lattice with a strong negation *.

Lemma 2.3 [4,8,9] *For each $x, y, z, x_i, y_i \in L$, we have the following properties.*

(1) *If $y \leq z$, $x \odot y \leq x \odot z$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.*

(2) $x \odot y \leq x \wedge y$.

(3) $x \rightarrow (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma} (x \rightarrow y_i)$.

(4) $(\bigvee_{i \in \Gamma} x_i) \rightarrow y = \bigwedge_{i \in \Gamma} (x_i \rightarrow y)$.

(5) $\bigwedge_{i \in \Gamma} y_i^* = (\bigvee_{i \in \Gamma} y_i)^*$ *and* $\bigvee_{i \in \Gamma} y_i^* = (\bigwedge_{i \in \Gamma} y_i)^*$.

(6) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

(7) $x \odot y = (x \rightarrow y^*)^*$ *and* $x \rightarrow y = y^* \rightarrow x^*$.

3 Join preserving maps and various concepts

Definition 3.1 Let X and Y be two sets. Let $\omega^{\rightarrow}, \phi^{\rightarrow}, \xi^{\rightarrow} : L^X \rightarrow L^Y$ and $\omega^{\leftarrow}, \phi^{\leftarrow}, \xi^{\leftarrow} : L^Y \rightarrow L^X$ be operators.

(1) The pair $(\phi^{\rightarrow}, \phi^{\leftarrow})$ is called an *isotone Galois connection* between X and Y if for $\mu \in L^X$ and $\rho \in L^Y$, $\phi^{\rightarrow}(\mu) \leq \rho$ iff $\mu \leq \phi^{\leftarrow}(\rho)$. Moreover, the

pair $(\xi^{\leftarrow}, \xi^{\rightarrow})$ is called an *isotone Galois connection* between X and Y if for $\mu \in L^X$ and $\rho \in L^Y$, $\xi^{\leftarrow}(\rho) \leq \mu$ iff $\rho \leq \xi^{\rightarrow}(\mu)$.

(2) The pair $(\omega^{\rightarrow}, \omega^{\leftarrow})$ is called *antitone Galois connection* between X and Y if for $\mu \in L^X$ and $\rho \in L^Y$, $\rho \leq \omega^{\rightarrow}(\mu)$ iff $\mu \leq \omega^{\leftarrow}(\rho)$.

Theorem 3.2 *Let $\phi^{\rightarrow} : L^X \to L^Y$ and $\phi^{\leftarrow} : L^Y \to L^X$ be operators. Let $(\phi^{\rightarrow}, \phi^{\leftarrow})$ be an isotone Galois connection between X and Y. For each $\mu, \mu_i \in L^X$ and $\rho, \rho_j \in L^Y$, the following properties hold:*

(1) *$\mu \leq \phi^{\leftarrow}(\phi^{\rightarrow}(\mu))$ and $\phi^{\rightarrow}(\phi^{\leftarrow}(\rho)) \leq \rho$.*

(2) *If $\mu_1 \leq \mu_2$, then $\phi^{\rightarrow}(\mu_1) \leq \phi^{\rightarrow}(\mu_2)$. Moreover, if $\rho_1 \leq \rho_2$, then $\phi^{\leftarrow}(\rho_1) \leq \phi^{\leftarrow}(\rho_2)$.*

(3) *$\phi^{\leftarrow}(\phi^{\rightarrow}(\phi^{\leftarrow}(\rho))) = \phi^{\leftarrow}(\rho)$ and $\phi^{\rightarrow}(\phi^{\leftarrow}(\phi^{\rightarrow}(\mu))) = \phi^{\rightarrow}(\mu)$.*

(4) *$\phi^{\rightarrow}(\bigvee_{i\in I} \mu_i) = \bigvee_{i\in I} \phi^{\rightarrow}(\mu_i)$ and $\phi^{\leftarrow}(\bigwedge_{j\in J} \rho_j) = \bigwedge_{j\in J} \phi^{\leftarrow}(\rho_j)$.*

(5) *If $\phi^{\leftarrow}(\phi^{\rightarrow}(\mu_i)) = \mu_i$, then $\phi^{\leftarrow}(\phi^{\rightarrow}(\bigwedge_{i\in I} \mu_i)) = \bigwedge_{i\in I} \mu_i$,*

(6) *If $(\phi^{\rightarrow}(\phi^{\leftarrow}(\rho_j)) = \rho_j$, then $\phi^{\rightarrow}(\phi^{\leftarrow}(\bigvee_{j\in J} \rho_j)) = \bigvee_{j\in J} \rho_j$.*

Proof. (1) Since $\phi^{\rightarrow}(\mu) \leq \phi^{\rightarrow}(\mu)$, we have $\mu \leq \phi^{\leftarrow}(\phi^{\rightarrow}(\mu))$. Since $\phi^{\leftarrow}(\rho) \leq \phi^{\leftarrow}(\rho)$, we have $\phi^{\rightarrow}(\phi^{\leftarrow}(\rho)) \leq \rho$.

(2) Since $\mu_1 \leq \mu_2 \leq \phi^{\leftarrow}(\phi^{\rightarrow}(\mu_2))$, $\phi^{\rightarrow}(\mu_1) \leq \phi^{\rightarrow}(\mu_2)$. Since $\phi^{\rightarrow}(\phi^{\leftarrow}(\rho_1)) \leq \rho_1 \leq \rho_2$, $\phi^{\leftarrow}(\rho_1) \leq \phi^{\leftarrow}(\rho_2)$.

(3) It easily proved from (1) and (2).

(4) By (2), $\phi^{\rightarrow}(\bigvee_{i\in I} \mu_i) \geq \bigvee_{i\in I} \phi^{\rightarrow}(\mu_i)$. Since $\phi^{\rightarrow}(\mu_i) \leq \bigvee_{i\in I} \phi^{\rightarrow}(\mu_i)$ implies $\mu_i \leq \phi^{\leftarrow}(\bigvee_{i\in I} \phi^{\rightarrow}(\mu_i))$, we have $\bigvee_{i\in I} \mu_i \leq \phi^{\leftarrow}(\bigvee_{i\in I} \phi^{\rightarrow}(\mu_i))$. Hence $\phi^{\rightarrow}(\bigvee_{i\in I} \mu_i) \leq \bigvee_{i\in I} \phi^{\rightarrow}(\mu_i)$.

(5) By (1), $\phi^{\leftarrow}(\phi^{\rightarrow}(\bigwedge \mu_i)) \geq \bigwedge \mu_i$ and by (2),

$$\bigwedge_{i\in I} \mu_i = \bigwedge_{i\in I} \phi^{\leftarrow}(\phi^{\rightarrow}(\mu_i)) \geq \phi^{\leftarrow}(\phi^{\rightarrow}(\bigwedge_{i\in I} \mu_i)).$$

Hence, $\phi^{\leftarrow}(\phi^{\rightarrow}(\bigwedge_{i\in I} \mu_i)) = \bigwedge_{i\in I} \mu_i$.

(6) It is similarly proved as (5).

Definition 3.3 Let $\omega^{\rightarrow}, \phi^{\rightarrow}, \xi^{\rightarrow} : L^X \to L^Y$ and $\omega^{\leftarrow}, \phi^{\leftarrow}, \xi^{\leftarrow} : L^Y \to L^X$ be functions. A pair $(\mu, \rho) \in L^X \times L^Y$ is called:

(1) *formal concept* if $\rho = \omega^{\rightarrow}(\mu)$ and $\mu = \omega^{\leftarrow}(\rho)$ where $(\omega^{\rightarrow}, \omega^{\leftarrow})$ is an antitone Galios connection,

(2) *an attribute oriented concept* if $\rho = \phi^{\rightarrow}(\mu)$ and $\mu = \phi^{\leftarrow}(\rho)$ where $(\phi^{\rightarrow}, \phi^{\leftarrow})$ is an isotone Galios connection,

(3) *an object oriented concept* if $\rho = \xi^{\rightarrow}(\mu)$ and $\mu = \xi^{\leftarrow}(\rho)$ where $(\xi^{\leftarrow}, \xi^{\rightarrow})$ is an isotone Galios connection.

Definition 3.4 An operator $\phi : L^X \to L^Y$ is called a join-generating operator, denoted by $\phi \in J(X, Y)$, if $\phi(\bigvee_{i\in\Gamma} \lambda_i) = \bigvee_{i\in\Gamma} \phi(\lambda_i)$, for $\{\lambda_i\}_{i\in\Gamma} \subset L^X$.

An operator $\psi : L^X \to L^Y$ is called a meet-generating operator, denoted by $\phi \in M(X, Y)$, if $\psi(\bigwedge_{i\in\Gamma} \lambda_i) = \bigwedge_{i\in\Gamma} \psi(\lambda_i)$, for $\{\lambda_i\}_{i\in\Gamma} \subset L^X$.

Theorem 3.5 *For $\phi^{\rightarrow} \in J(X,Y)$, Define functions $\omega_\phi^{\rightarrow}, \xi_\phi^{\rightarrow} : L^X \to L^Y$ and $\phi^{\leftarrow}, \omega_\phi^{\leftarrow}, \xi_\phi^{\leftarrow} : L^Y \to L^X$ as follows: , for all $\lambda \in L^X, \rho \in L^Y$,*

$$\phi^{\leftarrow}(\rho) = \bigvee\{\lambda \in L^X \mid \phi^{\rightarrow}(\lambda) \leq \rho\},$$

$$\omega_\phi^{\rightarrow}(\lambda) = (\phi^{\rightarrow}(\lambda))^*, \quad \omega_\phi^{\leftarrow}(\rho) = \phi^{\leftarrow}(\rho^*),$$

$$\xi_\phi^{\leftarrow}(\rho) = \bigwedge\{\lambda \in L^X \mid \phi^{\rightarrow}(\lambda^*) \leq \rho^*\}, \;\; \xi_\phi^{\rightarrow}(\lambda) = \bigvee\{\rho \in L^Y \mid \xi_\phi^{\leftarrow}(\rho) \leq \lambda\},$$

Then the following properties hold:

(1) The pair $(\phi^{\rightarrow}, \phi^{\leftarrow})$ is an isotone Galois connection and $(\phi^{\leftarrow}(\phi^{\rightarrow}(\lambda)), \phi^{\rightarrow}(\lambda))$ for each $\lambda \in L^X$ are attribute oriented concepts with $\phi^{\leftarrow} \in M(Y,X)$.

(2) If $\alpha \odot \phi^{\rightarrow}(\lambda) \leq \phi^{\rightarrow}(\alpha \odot \lambda)$ for $\lambda \in L^X$, then $\alpha \to \phi^{\leftarrow}(\rho) \leq \phi^{\leftarrow}(\alpha \to \rho)$ for $\rho \in L^Y$.

(3) The pair $(\omega_\phi^{\rightarrow}, \omega_\phi^{\leftarrow})$ is an antitone Galois connection and $(\omega_\phi^{\leftarrow}(\omega_\phi^{\rightarrow}(\lambda)), \omega_\phi^{\rightarrow}(\lambda))$ for $\lambda \in L^X$ are formal concepts.

(4) If $\omega_\phi^{\rightarrow}(\alpha \odot \mu) \leq \alpha \to \omega_\phi^{\rightarrow}(\mu)$, then $\omega_\phi^{\leftarrow}(\alpha \odot \rho) \geq \alpha \to \omega_\phi^{\leftarrow}(\rho)$.

(5) $\xi_\phi^{\leftarrow} \in J(Y,X)$ such that $\xi_\phi^{\leftarrow}(\rho) = (\phi^{\leftarrow}(\rho^*))^*$ and

$$\phi^{\rightarrow}(\lambda) \leq \rho \Leftrightarrow \lambda \leq \phi^{\leftarrow}(\rho) \Leftrightarrow \xi_\phi^{\leftarrow}(\rho^*) \leq \lambda^*$$

If $\alpha \odot \phi^{\rightarrow}(\lambda) \leq \phi^{\rightarrow}(\alpha \odot \lambda)$, then $\alpha \odot \xi_\phi^{\leftarrow}(\rho) \geq \xi_\phi^{\leftarrow}(\alpha \odot \rho)$.

(6) $\xi_\phi^{\rightarrow} \in M(X,Y)$ such that $\xi_\phi^{\rightarrow}(\lambda) = (\phi^{\rightarrow}(\lambda^*))^*$ and

$$\phi(\lambda) \leq \rho \Leftrightarrow \lambda \leq \phi^{\leftarrow}(\rho) \Leftrightarrow \xi_\phi^{\leftarrow}(\rho^*) \leq \lambda^* \Leftrightarrow \rho^* \leq \xi_\phi^{\rightarrow}(\lambda^*)$$

If $\alpha \odot \xi_\phi^{\leftarrow}(\rho) \geq \xi_\phi^{\leftarrow}(\alpha \odot \rho)$, then $\alpha \to \xi_\phi^{\rightarrow}(\lambda) \geq \xi_\phi^{\rightarrow}(\alpha \to \lambda)$.

(7) The pair $(\xi_\phi^{\leftarrow}, \xi_\phi^{\rightarrow})$ is an isotone Galois connection and $(\xi_\phi^{\leftarrow}(\rho), \xi_\phi^{\rightarrow}(\xi_\phi^{\leftarrow}(\rho)))$ for each $\rho \in L^Y$, are object oriented concepts.

Proof. (1) Since ϕ is a join preserving map and $\phi^{\leftarrow}(\rho) = \bigvee\{\lambda \in L^X \mid \phi^{\rightarrow}(\lambda) \leq \rho\}$, we have

$$\phi^{\rightarrow}(\lambda) \leq \rho \Leftrightarrow \lambda \leq \phi^{\leftarrow}(\rho).$$

Hence $(\phi^{\rightarrow}, \phi^{\leftarrow})$ is an isotone Galois connection and, by Theorem 3.2(3), $(\phi^{\leftarrow}(\phi^{\rightarrow}(\lambda)), \phi^{\rightarrow}(\lambda))$ are attribute oriented concepts. Moreover, $\phi^{\leftarrow}(\bigwedge_{i\in\Gamma} \rho_i) = \bigwedge_{i\in\Gamma} \phi^{\leftarrow}(\rho_i)$ from

$$\begin{array}{ll} \bigwedge_{i\in\Gamma} \phi^{\leftarrow}(\rho_i) \geq \lambda & \Leftrightarrow \phi^{\leftarrow}(\rho_i) \geq \lambda, \;\; \forall i \in \Gamma \Leftrightarrow \phi^{\rightarrow}(\lambda) \leq \rho_i, \;\; \forall i \in \Gamma \\ & \Leftrightarrow \phi^{\rightarrow}(\lambda) \leq \bigwedge_{i\in\Gamma} \rho_i, \Leftrightarrow \phi^{\leftarrow}(\bigwedge_{i\in\Gamma} \rho_i) \geq \lambda. \end{array}$$

(2) Let $\alpha \odot \phi^{\rightarrow}(\lambda) \leq \phi^{\rightarrow}(\alpha \odot \lambda)$ for $\lambda \in L^X$. For $\mu \leq \alpha \to \phi^{\leftarrow}(\rho)$, $\mu \odot \alpha \leq \phi^{\leftarrow}(\rho)$ iff $\phi^{\rightarrow}(\mu \odot \alpha) \leq \rho$. Then $\alpha \odot \phi^{\rightarrow}(\mu) \leq \rho$ iff $\phi^{\rightarrow}(\mu) \leq \alpha \to \rho$ iff $\mu \leq \phi^{\leftarrow}(\alpha \to \rho)$. Hence $\alpha \to \phi^{\leftarrow}(\rho) \leq \phi^{\leftarrow}(\alpha \to \rho)$.

(3) It follows from $\omega_\phi^{\rightarrow}(\omega_\phi^{\leftarrow}(\omega_\phi^{\rightarrow}(\lambda))) = \omega_\phi^{\rightarrow}(\omega_\phi^{\leftarrow}(\phi^{\rightarrow}(\lambda)^*)) = \omega_\phi^{\rightarrow}(\phi^{\leftarrow}(\phi^{\rightarrow}(\lambda))) = (\phi^{\rightarrow}(\phi^{\leftarrow}(\phi^{\rightarrow}(\lambda))))^* = (\phi^{\rightarrow}(\lambda))^* = \omega_\phi^{\rightarrow}(\lambda)$ and

$$\rho \le \omega_\phi^{\rightarrow}(\lambda) \Leftrightarrow \rho \le (\phi^{\rightarrow}(\lambda))^* \Leftrightarrow \phi^{\rightarrow}(\lambda) \le \rho^* \Leftrightarrow \lambda \le \phi^{\leftarrow}(\rho^*) = \omega_\phi^{\leftarrow}(\rho).$$

(4) Let $\omega_\phi^{\rightarrow}(\alpha \odot \mu) = (\phi^{\rightarrow}(\alpha \odot \mu))^* \le \alpha \to \omega_\phi^{\rightarrow}(\mu) = \alpha \to (\phi^{\rightarrow}(\mu))^* = (\alpha \odot \phi^{\rightarrow}(\mu))^*$ from Lemma 2.3(7). So, $\alpha \odot \phi^{\rightarrow}(\lambda) \le \phi^{\rightarrow}(\alpha \odot \lambda)$. By (2), $\alpha \to \phi^{\leftarrow}(\rho) \le \phi^{\leftarrow}(\alpha \to \rho)$.

$\omega_\phi^{\leftarrow}(\alpha \odot \rho) = \phi^{\leftarrow}((\alpha \odot \rho)^*) = \phi^{\leftarrow}(\alpha \to \rho^*) \ge \alpha \to \phi^{\leftarrow}(\rho^*) = \alpha \to \omega_\phi^{\leftarrow}(\rho)$.

(5) By Lemma 2.3(5), we have

$$\begin{aligned}\xi_\phi^{\leftarrow}(\rho) &= \bigwedge\{\lambda \in L^X \mid \phi^{\rightarrow}(\lambda^*) \le \rho^*\} \\ &= \Big(\bigvee\{\lambda^* \in L^X \mid \lambda^* \le \phi^{\leftarrow}(\rho^*)\}\Big)^* \quad = (\phi^{\leftarrow}(\rho^*))^*.\end{aligned}$$

It follows $\xi_\phi^{\leftarrow} \in J(Y,X)$ and $\phi^{\rightarrow}(\lambda) \le \rho \Leftrightarrow \lambda \le \phi^{\leftarrow}(\rho) \Leftrightarrow \xi_\phi^{\leftarrow}(\rho^*) \le \lambda^*$. By (2) and Lemma 2.3(7), we have:

$$\begin{aligned}\xi_\phi^{\leftarrow}(\alpha \odot \rho) &= (\phi^{\leftarrow}((\alpha \odot \rho)^*))^* = (\phi^{\leftarrow}(\alpha \to \rho^*))^* \\ &\le (\alpha \to \phi^{\leftarrow}(\rho^*))^* = \alpha \odot (\phi^{\leftarrow}(\rho^*))^* \\ &= \alpha \odot \xi_\phi^{\leftarrow}(\rho).\end{aligned}$$

(6)

$$\begin{aligned}\xi_\phi^{\rightarrow}(\lambda) &= \bigvee\{\rho \in L^Y \mid \xi_\phi^{\leftarrow}(\rho) \le \lambda\} \\ &= \bigvee\{\rho \in L^Y \mid \phi^{\rightarrow}(\lambda^*) \le \rho^*\} = (\phi(\lambda^*))^*.\end{aligned}$$

$\xi_\phi^{\rightarrow} \in M(X,Y)$ and other cases are similarly proved as (1) and (5).

(7) $\xi_\phi^{\leftarrow}(\xi_\phi^{\rightarrow}(\xi_\phi^{\leftarrow}(\rho))) = \xi_\phi^{\leftarrow}(\xi_\phi^{\rightarrow}(\phi^{\leftarrow}(\rho^*))^*) = \xi_\phi^{\leftarrow}((\phi^{\rightarrow}(\phi^{\leftarrow}(\rho^*)))^*)$
$= (\phi^{\leftarrow}(\phi^{\rightarrow}(\phi^{\leftarrow}(\rho^*))))^* = (\phi^{\leftarrow}(\rho^*))^* = \xi_\phi^{\leftarrow}(\rho)$. Other cases are easily proved.

Theorem 3.6 *Let X and Y be sets and $R \in L^{X \times Y}$. Define a function $\phi_R^{\rightarrow} : L^X \to L^Y$ as $\phi_R^{\rightarrow}(\lambda)(y) = \bigvee_{x \in X}(\lambda(x) \odot R(x,y))$. Then we have the following properties:*

(1) *the pair $(\phi_R^{\rightarrow}, \phi_R^{\leftarrow})$ is an isotone Galois connection which $(\phi_R^{\leftarrow}(\phi_R^{\rightarrow}(\lambda)), \phi_R^{\rightarrow}(\lambda))$, for each $\lambda \in L^X$, are attribute oriented concepts such that*

$$\phi_R^{\leftarrow}(\rho)(x) = \bigwedge_{y \in Y}(R(x,y) \to \rho(y)).$$

(2) $\alpha \odot \phi_R^{\rightarrow}(\lambda) = \phi_R^{\rightarrow}(\alpha \odot \lambda)$, $\alpha \to \phi_R^{\leftarrow}(\rho) = \phi_R^{\leftarrow}(\alpha \to \rho) = \phi_{\alpha \odot R}^{\leftarrow}(\rho)$.

(3) *The pair $(\xi_{\phi_R}^{\leftarrow}, \xi_{\phi_R}^{\rightarrow})$ is an isotone Galois connection which $(\xi_{\phi_R}^{\leftarrow}(\rho), \xi_{\phi_R}^{\rightarrow}(\xi_{\phi_R}^{\leftarrow}(\rho)))$, for each $\rho \in L^X$, are object oriented concepts with*

$$\xi_{\phi_R}^{\leftarrow}(\rho)(x) = \bigvee_{y \in X}(R(x,y) \odot \rho(y)), \quad \xi_{\phi_R}^{\rightarrow}(\mu)(y) = \bigwedge_{x \in X}(R(x,y) \to \mu(x)).$$

(4) $\omega^{\rightarrow}_{\phi_R}(\mu) = (\phi^{\rightarrow}_R(\mu))^* = \xi^{\rightarrow}_{\phi_R}(\mu^*)$, $\omega^{\leftarrow}_{\phi_R}(\rho) = \phi^{\leftarrow}_R(\rho^*) = (\xi^{\leftarrow}_{\phi_R}(\rho))^*$, $\omega^{\rightarrow}_{\phi_R}(\alpha \odot \mu) = \alpha \to \omega^{\rightarrow}_{\phi_R}(\mu)$ *and* $\omega^{\leftarrow}_{\phi_R}(\alpha \odot \rho) = \alpha \to \omega^{\leftarrow}_{\phi_R}(\rho)$ *where*

$$\omega^{\rightarrow}_{\phi_R}(\mu)(y) = \bigwedge_{y \in Y} (\mu(x) \to R^*(x,y)), \ \ \omega^{\leftarrow}_{\phi_R}(\rho)(x) = \bigwedge_{y \in Y} (\rho(y) \to R^*(x,y)).$$

(5) *The pair* $(\omega^{\rightarrow}_{\phi_R}, \omega^{\leftarrow}_{\phi_R})$ *is an antitone Galois connection where* $(\omega^{\leftarrow}_{\phi_R}(\omega^{\rightarrow}_{\phi_R}(\lambda)), \omega^{\rightarrow}_{\phi_R}(\lambda))$ *for each* $\lambda \in L^X$ *are attribute concepts.*

Proof. (1) $\phi^{\rightarrow}_R$ is join preserving because

$$\begin{aligned} \phi^{\rightarrow}_R(\bigvee_i \lambda_i)(y) &= \bigvee_{x \in X}(\bigvee_i \lambda_i(x)) \odot R(x,y) \\ &= \bigvee_i(\bigvee_{x \in X} \lambda_i(x) \odot R(x,y)) \\ &= \bigvee_i \phi^{\rightarrow}_R(\lambda_i)(y). \end{aligned}$$

By Theorem 3.5, we obtain $\phi^{\leftarrow}_R$ as follows

$$\begin{aligned} \phi^{\leftarrow}_R(\rho)(x) &= \bigvee\{\lambda(x) \in L^X \mid \phi^{\rightarrow}_R(\lambda) \le \rho\} \\ &= \bigvee\{\lambda(x) \in L^X \mid \lambda(x) \le \bigwedge(R(x,y) \to \rho(y))\} \\ &= \bigwedge(R(x,y) \to \rho(y)) \end{aligned}$$

Thus, $(\phi^{\rightarrow}_R, \phi^{\leftarrow}_R)$ is an isotone Galois connection.

(2) From Lemma 2.3(3,6),

$$\begin{aligned} \phi^{\leftarrow}_R(\alpha \to \rho)(x) &= \bigwedge_{y \in Y}(R(x,y) \to (\alpha \to \rho(y))) \\ &= \bigwedge_{y \in Y}(\alpha \to (R(x,y) \to \rho(y))) \\ &= \alpha \to \bigwedge_{y \in Y}(R(x,y) \to \rho(y)) = \alpha \to \phi^{\leftarrow}_R(\rho)(x) \\ &= \bigwedge_{y \in Y}(\alpha \odot R(x,y) \to \rho(y)) = \phi^{\leftarrow}_{\alpha \odot R}(\rho)(x). \end{aligned}$$

(3) From Theorem 3.5(5) and Lemma 2.3(6),

$$\begin{aligned} \phi^{\rightarrow}_R(\lambda^*)(y) \le \rho^*(y) \quad &\Leftrightarrow \bigvee_{x \in X} R(x,y) \odot \lambda^*(x) \le \rho^*(y) \\ \Leftrightarrow \lambda^*(x) \le R(x,y) \to \rho^*(y) \quad &\Leftrightarrow \lambda(x) \ge \bigvee_{y \in Y} R(x,y) \odot \rho(y). \end{aligned}$$

It follows

$$\xi^{\leftarrow}_{\phi_R}(\rho)(x) = \bigwedge\{\lambda(x) \mid \phi^{\rightarrow}_R(\lambda^*) \le \rho^*\} = \bigvee_{y \in Y} R(x,y) \odot \rho(y)$$

Since $\xi^{\rightarrow}_{\phi_R}(\lambda) = \left(\phi^{\rightarrow}_R(\lambda^*)\right)^*$ from Theorem 3.5(6), by Lemma 2.3(6,7), we have from:

$$\begin{aligned} \xi^{\rightarrow}_{\phi_R}(\lambda) &= \left(\phi^{\rightarrow}_R(\lambda^*)\right)^* = \left(\bigvee_{x \in X}(R(x,y) \odot \lambda^*(x))\right)^* \\ &= \bigwedge_{x \in X}(R(x,y) \odot \lambda^*(x))^* = \bigwedge_{x \in X}(R(x,y) \to \lambda(x)). \end{aligned}$$

Other cases follows from Theorem 3.5 (5-7).

(4)

$$\begin{aligned}\omega^{\rightarrow}_{\phi_R}(\mu)(y) &= (\phi^{\rightarrow}_R(\mu))^*(y) = (\bigvee_{x\in X}\mu(x)\odot R(x,y))^* \\ &= \bigwedge_{x\in X}(\mu(x)\odot R(x,y))^* \\ &= \bigwedge_{x\in X}(R(x,y)\to \mu^*(x) = \xi^{\rightarrow}_{\phi_R}(\mu^*)(y) \\ &= \bigwedge_{x\in X}(\mu(x)\to R^*(x,y)) \ \text{(by Lemma 2.3(7)).}\end{aligned}$$

$$\begin{aligned}\omega^{\leftarrow}_{\phi_R}(\rho)(x) &= \phi^{\leftarrow}_R(\rho^*)(x) = (\bigvee_{y\in Y}(R(x,y)\to \rho^*(y)) \\ &= (\bigvee_{y\in Y}(\rho(y)\odot R(x,y)))^* = (\xi^{\leftarrow}_R(\rho))^* \\ &= \bigwedge_{x\in X}(\rho(y)\to R^*(x,y)) \ \text{(by Lemma 2.3(7)).}\end{aligned}$$

Other cases follows from Theorem 3.5 (3).

Example 3.7 Let $X = \{a,b\}$ and $Y = \{u,v,w\}$ be sets. Let $(L = \{0,\frac{1}{2},1\},\odot)$ be a complete residuated lattice such that $x\odot y = (x+y-1)\vee 1$, $x\to y = (1-x+y)\wedge 1$, $x^* = 1-x$ and $R\in L^{X\times Y}$ as

$$R = \begin{pmatrix} 1 & \frac{1}{2} & 1 \\ 1 & 0 & \frac{1}{2} \end{pmatrix}$$

For $\lambda\in L^X, \rho\in L^Y$, we denote $(\lambda(a),\lambda(b))$ and $(\rho(u),\rho(v),\rho(w))$, we obtain;

$$\begin{pmatrix} \phi^{\rightarrow}_R(0,0)=(0,0,0) & \phi^{\rightarrow}_R(0,\frac{1}{2})=(\frac{1}{2},0,0) & \phi^{\rightarrow}_R(0,1)=(1,0,\frac{1}{2}) \\ \phi^{\rightarrow}_R(\frac{1}{2},0)=(\frac{1}{2},0,\frac{1}{2}) & \phi^{\rightarrow}_R(\frac{1}{2},\frac{1}{2})=(\frac{1}{2},0,\frac{1}{2}) & \phi^{\rightarrow}_R(\frac{1}{2},1)=(1,0,\frac{1}{2}) \\ \phi^{\rightarrow}_R(1,0)=(1,\frac{1}{2},1) & \phi^{\rightarrow}_R(1,\frac{1}{2})=(1,\frac{1}{2},1) & \phi^{\rightarrow}_R(1,1)=(1,\frac{1}{2},1) \end{pmatrix}$$

$$\begin{pmatrix} \phi^{\leftarrow}_R(0,0,0)=(0,0) & \phi^{\leftarrow}_R(0,0,\frac{1}{2})=(0,0) & \phi^{\leftarrow}_R(0,0,1)=(0,0) \\ \phi^{\leftarrow}_R(0,\frac{1}{2},0)=(0,0) & \phi^{\leftarrow}_R(0,\frac{1}{2},\frac{1}{2})=(0,0) & \phi^{\leftarrow}_R(0,\frac{1}{2},\frac{1}{2})=(0,0) \\ \phi^{\leftarrow}_R(0,1,0)=(0,0) & \phi^{\leftarrow}_R(0,1,\frac{1}{2})=(0,0) & \phi^{\leftarrow}_R(0,1,1)=(0,0) \\ \phi^{\leftarrow}_R(\frac{1}{2},0,0)=(0,\frac{1}{2}) & \phi^{\leftarrow}_R(\frac{1}{2},0,\frac{1}{2})=(\frac{1}{2},\frac{1}{2}) & \phi^{\leftarrow}_R(\frac{1}{2},0,1)=(\frac{1}{2},\frac{1}{2}) \\ \phi^{\leftarrow}_R(\frac{1}{2},\frac{1}{2},0)=(0,\frac{1}{2}) & \phi^{\leftarrow}_R(\frac{1}{2},\frac{1}{2},\frac{1}{2})=(\frac{1}{2},\frac{1}{2}) & \phi^{\leftarrow}_R(\frac{1}{2},\frac{1}{2},1)=(\frac{1}{2},\frac{1}{2}) \\ \phi^{\leftarrow}_R(\frac{1}{2},1,0)=(0,\frac{1}{2}) & \phi^{\leftarrow}_R(\frac{1}{2},1,\frac{1}{2})=(\frac{1}{2},\frac{1}{2}) & \phi^{\leftarrow}_R(\frac{1}{2},1,1)=(\frac{1}{2},\frac{1}{2}) \\ \phi^{\leftarrow}_R(1,0,0)=(0,\frac{1}{2}) & \phi^{\leftarrow}_R(1,0,\frac{1}{2})=(\frac{1}{2},1) & \phi^{\leftarrow}_R(1,0,1)=(\frac{1}{2},\frac{1}{2}) \\ \phi^{\leftarrow}_R(1,\frac{1}{2},0)=(0,\frac{1}{2}) & \phi^{\leftarrow}_R(1,\frac{1}{2},\frac{1}{2})=(\frac{1}{2},1) & \phi^{\leftarrow}_R(1,\frac{1}{2},1)=(1,1) \\ \phi^{\leftarrow}_R(1,1,0)=(0,\frac{1}{2}) & \phi^{\leftarrow}_R(1,1,\frac{1}{2})=(\frac{1}{2},1) & \phi^{\leftarrow}_R(1,1,1)=(1,1) \end{pmatrix}$$

We obtain an attribute oriented concept family as follows:

$$\{((0,0),(0,0,0)),((0,\tfrac{1}{2}),(\tfrac{1}{2},0,0)),((\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},0,\tfrac{1}{2})), \\ ((\tfrac{1}{2},1),(1,0,\tfrac{1}{2})),((1,1),(1,\tfrac{1}{2},1))\}$$

$$\begin{pmatrix} \xi^{\leftarrow}_R(0,0,0)=(0,0) & \xi^{\leftarrow}_R(0,0,\frac{1}{2})=(\frac{1}{2},0) & \xi^{\leftarrow}_R(0,0,1)=(1,\frac{1}{2}) \\ \xi^{\leftarrow}_R(0,\frac{1}{2},0)=(0,0) & \xi^{\leftarrow}_R(0,\frac{1}{2},\frac{1}{2})=(\frac{1}{2},0) & \xi^{\leftarrow}_R(0,\frac{1}{2},1)=(1,\frac{1}{2}) \\ \xi^{\leftarrow}_R(0,1,0)=(\frac{1}{2},0) & \xi^{\leftarrow}_R(0,1,\frac{1}{2})=(\frac{1}{2},0) & \xi^{\leftarrow}_R(0,1,1)=(1,\frac{1}{2}) \\ \xi^{\leftarrow}_R(\frac{1}{2},0,0)=(\frac{1}{2},\frac{1}{2}) & \xi^{\leftarrow}_R(\frac{1}{2},0,\frac{1}{2})=(\frac{1}{2},\frac{1}{2}) & \xi^{\leftarrow}_R(\frac{1}{2},0,1)=(1,\frac{1}{2}) \\ \xi^{\leftarrow}_R(\frac{1}{2},\frac{1}{2},0)=(\frac{1}{2},\frac{1}{2}) & \xi^{\leftarrow}_R(\frac{1}{2},\frac{1}{2},\frac{1}{2})=(\frac{1}{2},\frac{1}{2}) & \xi^{\leftarrow}_R(\frac{1}{2},\frac{1}{2},1)=(1,\frac{1}{2}) \\ \xi^{\leftarrow}_R(\frac{1}{2},1,0)=(\frac{1}{2},\frac{1}{2}) & \xi^{\leftarrow}_R(\frac{1}{2},1,\frac{1}{2})=(\frac{1}{2},\frac{1}{2}) & \xi^{\leftarrow}_R(\frac{1}{2},1,1)=(1,\frac{1}{2}) \\ \xi^{\leftarrow}_R(1,0,0)=(1,1) & \xi^{\leftarrow}_R(1,0,\frac{1}{2})=(1,1) & \xi^{\leftarrow}_R(1,0,1)=(1,1) \\ \xi^{\leftarrow}_R(1,\frac{1}{2},0)=(1,1) & \xi^{\leftarrow}_R(1,\frac{1}{2},\frac{1}{2})=(1,1) & \xi^{\leftarrow}_R(1,\frac{1}{2},1)=(1,1) \\ \xi^{\leftarrow}_R(1,1,0)=(1,1) & \xi^{\leftarrow}_R(1,1,\frac{1}{2})=(1,1) & \xi^{\leftarrow}_R(1,1,1)=(1,1) \end{pmatrix}$$

$$\begin{pmatrix} \xi^{\rightarrow}_R(0,0)=(0,\frac{1}{2},0) & \xi^{\rightarrow}_R(0,\frac{1}{2})=(0,\frac{1}{2},0) & \xi^{\rightarrow}_R(0,1)=(0,\frac{1}{2},0) \\ \xi^{\rightarrow}_R(\frac{1}{2},0)=(0,1,\frac{1}{2}) & \xi^{\rightarrow}_R(\frac{1}{2},\frac{1}{2})=(\frac{1}{2},1,\frac{1}{2}) & \xi^{\rightarrow}_R(\frac{1}{2},1)=(\frac{1}{2},1,\frac{1}{2}) \\ \xi^{\rightarrow}_R(1,0)=(0,1,\frac{1}{2}) & \xi^{\rightarrow}_R(1,\frac{1}{2})=(\frac{1}{2},1,1) & \xi^{\rightarrow}_R(1,1)=(1,1,1) \end{pmatrix}$$

We obtain an object oriented concept family as follows:

$$\{((0,0),(0,\frac{1}{2},0)),((\frac{1}{2},0),(0,1,\frac{1}{2})),((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},1,\frac{1}{2})),$$
$$((1,\frac{1}{2}),(\frac{1}{2},1,1)),((1,1),(1,1,1))\}$$

$$R^* = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} \omega^{\rightarrow}_{\phi_R}(0,0)=(1,1,1) & \omega^{\rightarrow}_{\phi_R}(0,\frac{1}{2})=(\frac{1}{2},1,1) & \omega^{\rightarrow}_{\phi_R}(0,1)=(0,1,\frac{1}{2}) \\ \omega^{\rightarrow}_{\phi_R}(\frac{1}{2},0)=(\frac{1}{2},1,\frac{1}{2}) & \omega^{\rightarrow}_{\phi_R}(\frac{1}{2},\frac{1}{2})=(\frac{1}{2},1,\frac{1}{2}) & \omega^{\rightarrow}_{\phi_R}(\frac{1}{2},1)=(0,1,\frac{1}{2}) \\ \omega^{\rightarrow}_{\phi_R}(1,0)=(0,\frac{1}{2},0) & \omega^{\rightarrow}_{\phi_R}(1,\frac{1}{2})=(0,\frac{1}{2},0) & \omega^{\rightarrow}_{\phi_R}(1,1)=(0,\frac{1}{2},0) \end{pmatrix}$$

$$\begin{pmatrix} \omega^{\leftarrow}_{\phi_R}(1,1,1)=(0,0) & \omega^{\leftarrow}_{\phi_R}(1,1,\frac{1}{2})=(0,0) & \omega^{\leftarrow}_{\phi_R}(1,1,0)=(0,0) \\ \omega^{\leftarrow}_{\phi_R}(1,\frac{1}{2},1)=(0,0) & \omega^{\leftarrow}_{\phi_R}(1,\frac{1}{2},\frac{1}{2})=(0,0) & \omega^{\leftarrow}_{\phi_R}(1,\frac{1}{2},0)=(0,0) \\ \omega^{\leftarrow}_{\phi_R}(1,0,1)=(0,0) & \omega^{\leftarrow}_{\phi_R}(1,0,\frac{1}{2})=(0,0) & \omega^{\leftarrow}_{\phi_R}(1,0,0)=(0,0) \\ \omega^{\leftarrow}_{\phi_R}(\frac{1}{2},1,1)=(0,\frac{1}{2}) & \omega^{\leftarrow}_{\phi_R}(\frac{1}{2},1,\frac{1}{2})=(\frac{1}{2},\frac{1}{2}) & \omega^{\leftarrow}_{\phi_R}(\frac{1}{2},1,0)=(\frac{1}{2},\frac{1}{2}) \\ \omega^{\leftarrow}_{\phi_R}(\frac{1}{2},\frac{1}{2},1)=(0,\frac{1}{2}) & \omega^{\leftarrow}_{\phi_R}(\frac{1}{2},\frac{1}{2},\frac{1}{2})=(\frac{1}{2},\frac{1}{2}) & \omega^{\leftarrow}_{\phi_R}(\frac{1}{2},\frac{1}{2},0)=(\frac{1}{2},\frac{1}{2}) \\ \omega^{\leftarrow}_{\phi_R}(\frac{1}{2},0,1)=(0,\frac{1}{2}) & \omega^{\leftarrow}_{\phi_R}(\frac{1}{2},0,\frac{1}{2})=(\frac{1}{2},\frac{1}{2}) & \omega^{\leftarrow}_{\phi_R}(\frac{1}{2},0,0)=(\frac{1}{2},\frac{1}{2}) \\ \omega^{\leftarrow}_{\phi_R}(0,1,1)=(0,\frac{1}{2}) & \omega^{\leftarrow}_{\phi_R}(0,1,\frac{1}{2})=(\frac{1}{2},1) & \omega^{\leftarrow}_{\phi_R}(0,1,0)=(\frac{1}{2},\frac{1}{2}) \\ \omega^{\leftarrow}_{\phi_R}(0,\frac{1}{2},1)=(0,\frac{1}{2}) & \omega^{\leftarrow}_{\phi_R}(0,\frac{1}{2},\frac{1}{2})=(\frac{1}{2},1) & \omega^{\leftarrow}_{\phi_R}(0,\frac{1}{2},0)=(1,1) \\ \omega^{\leftarrow}_{\phi_R}(0,0,1)=(0,\frac{1}{2}) & \omega^{\leftarrow}_{\phi_R}(0,0,\frac{1}{2})=(\frac{1}{2},1) & \omega^{\leftarrow}_{\phi_R}(0,0,0)=(1,1) \end{pmatrix}$$

We obtain a formal concept family as follows:

$$\{((0,0),(1,1,1)),((0,\frac{1}{2}),(\frac{1}{2},1,1)),((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},1,\frac{1}{2})),$$
$$((\frac{1}{2},1),(0,1,\frac{1}{2})),((1,1),(0,\frac{1}{2},0))\}$$

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