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The Probability of Default Implied by Binary Option Prices

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Abstract

The maximum entropy method which is one of non-parametric methods is used to estimate the probability of default. From binary option prices as the constraints, we explicitly find it while Capuano(2008) suggests numerical computation from European call option prices.

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1 Introduction

The maximum entropy principle has use in the literature of finance, especially derivatives [1, 2, 3, 4, 5, 9, 10, 11, 12, 13]. The purpose of this paper is to estimate the probability of default from binary option prices using the maximum entropy method [8]. There are three difference in our approach. First, while previous papers primary dealt with vanilla options to find the maximum entropy distribution(**MED**), we also use binary options and multi-asset options prices [6]. Second, we suggest the modified MED since the MED does not exist in binary options. Third, while the Jacobian matrix is represented as the covariance matrix is used to compute Lagrange multipliers in general, we suggest a method that find them sequentially.

The present paper is organized as follows. We begin Section 2 with a review of the MED under binary option prices constraints Moreover we introduce a modified MED and the probability of default. Section 3 gives the results for binary options. We conclude in Section 4.

2 The Probability of Default and Quasi-Implied Distributions

For $0 = K_0 < K_1 < \cdots < K_n < \infty$, let X be a continuous random variable that represents the price of an asset at a maturity time T. Also let V_i be the price of an option with underlying asset X, maturity T and strike price K_i, $i = 1, \ldots, n$ and let V_L be be the price of an option with underlying asset X, maturity T and strike price L.

Definition 2.1. *(i) Let X be continuous random variables with probability density $\rho(x)$, $x \in [0, \infty)$. The differential entropy of X is defined by*

$$H(X) = -\int_0^\infty \rho(x) \ln \rho(x) dx.$$

We also write $H(\rho)$ for the above quantity [8].

(ii) The probability of default(PoD) of X is defined by

$$PoD(U) = \int_0^U \rho(x) dx$$

where U is the default threshold of X which is the value that triggers the default [6].

(iii) Suppose that $\rho \geq 0$ satisfies the probability constraint

$$\int_0^\infty \rho(x) dx = 1. \tag{1}$$

Also we assume constraints which are expressed in the form

$$\mathbb{E}[c_i(X)] = \int_0^\infty v_i(x)\rho(x) dx = V_i, \tag{2}$$

*where $v_i(x)$ is the ith discounted option payoff with strike price K_i for $i = 1, \ldots, n$. The distribution that maximizes $H(\rho)$ under the constraints (1),(2) is called an implied probability density function(**pdf**).*

(iv) Assume that the domain of X is restricted from $[0, \infty)$ to $[0, L]$ for a sufficiently large L, i.e., the constraints (1),(2) change to

$$\int_0^L \hat{\rho}(x) dx = 1, \tag{3}$$

$$\int_0^L v_i(x)\hat{\rho}(x) dx = V_i. \tag{4}$$

The distribution that maximizes $H(\hat{\rho})$ under the constraints (3),(4) is called a quasi-implied pdf [7].

Remark 2.2. *(i) The Rényi entropy of X with order $\alpha > 0$, $\alpha \neq 1$, is defined by*

$$H_\alpha(X) = \frac{1}{1-\alpha} \ln \left(\int_0^\infty \rho(x)^\alpha dx \right).$$

Note that $\lim_{\alpha \to 1} H_\alpha(\rho) = H(\rho)$ *and for* $N = 1$, $H(\rho) = -\int_0^\infty \rho(x) \ln \rho(x) dx$.

(ii) There is no guarantee that the implied pdf always exists in general options. For example, the implied pdf does not exist in cash-or-nothings, asset-or-nothing put options and etc. Thus we introduced the quasi-implied pdf.

Lemma 2.3. *(i) The implied pdf is given by*

$$\rho(x) = \frac{1}{Q} \exp\left(\sum_{i=1}^n \lambda_i v_i(x) \right).$$

where

$$Q = \int_0^\infty \exp\left(\sum_{i=1}^n \lambda_i v_i(x) \right) dx$$

and $\lambda_1, \ldots, \lambda_n$ *is chosen so that* ρ *satisfies the constraints (1),(2).*

(ii) The quasi-implied pdf is given by

$$\hat{\rho}(x) = \frac{1}{\hat{Q}} \exp\left(\sum_{i=1}^n \hat{\lambda}_i v_i(x) \right)$$

where

$$\hat{Q} = \int_0^L \exp\left(\sum_{i=1}^n \hat{\lambda}_i v_i(x) \right) dx$$

and $\hat{\lambda}_1, \ldots, \hat{\lambda}_n$ *are chosen so that* $\hat{\rho}$ *satisfies the constraints (3),(4).*

Proof. (i) Consult [5, 8]. (ii) The difference between (i) and (ii) is whether a support of X is closed or not. □

We close this section with introducing the quasi-implied PoD. Capuano [6] considers the following problem.

$$\min_D \{ \min_{\rho(x)} \int_0^\infty \rho(x) \log \frac{\rho(x)}{\rho^0(x)} dx \}$$

where D is the debt of underlying asset X and $\rho^0(x)$ is the prior probability density function of X, the posterior density $\rho(x)$. $\rho(x) \log \frac{\rho(x)}{\rho^0(x)}$ is the cross-entropy between $\rho(x)$ and $\rho^0(x)$. Hence he obtained an implied PoD from the above problem.

On the other hand, we use Shannon entropy and the quasi-implied pdf, i.e., our problem is as follows:

$$\max_{D}\{\max_{\hat{\rho}(x)} H(\hat{\rho})\} \tag{5}$$

A quasi-implied PoD is obtained from (5).

Definition 2.4. *A quasi-implied PoD is defined by*

$$\int_0^D \hat{\rho}(x)dx$$

where D and $\hat{\rho}(x)$ are the solution of the problem (5).

3 Application to Binary Options

We apply our problem to binary options, especially cash-or-nothing call options. For a simplicity we set discount factor being equal to 1 as in [2, 5]. Now we consider cash-or-nothing call option whose payoff $v_i(x)$ is $\mathbf{1}_{x\geq D+K_i}$. Hence

$$V_i = \int_0^\infty \mathbf{1}_{\{x\geq D+K_i\}}\rho(x)dx = \int_{D+K_i}^\infty \rho(x)dx$$

and

$$V_i = \int_{D+K_i}^{L} \hat{\rho}(x)dx.$$

Theorem 3.1. *(i) The implied pdf for a cash-or-nothing call does not exist.*
(ii) The quasi-implied pdf for a cash-or-nothing call is given by

$$\hat{\rho}(x) = \begin{cases} \frac{1-V_0}{D}, & 0 \leq x < D, \\ \frac{V_i - V_{i+1}}{K_{i+1}-K_i}, & D+K_i \leq x < D+K_{i+1} \ \textit{for } i = 0, \dots, n-1, \\ \frac{V_n}{L-D-K_n}, & D+K_n \leq x < L. \end{cases}$$

(iii) The optimal solution D of the problem (5) is

$$L - K_n - \frac{V_n}{1-V_0}$$

and the quasi-implied PoD is

$$1 - V_0.$$

Proof. (i) Since the denominator Q diverges, it does not exist [7].

(ii) From Lemma 2.3(ii), we know

$$\hat{\rho}(x) = \frac{1}{\hat{Q}} \exp\left(\sum_{i=1}^{n} \hat{\lambda}_i \mathbf{1}_{x \geq D+K_i} \right)$$

where

$$\hat{Q} = \int_0^L \exp\left(\sum_{i=1}^{n} \hat{\lambda}_i \mathbf{1}_{x \geq D+K_i} \right) dx$$

and $\hat{\lambda}_1, \ldots, \hat{\lambda}_n$ are chosen so that $\hat{\rho}$ satisfies the constraints (3),(4). Clearly we know the probability in each subinterval is constant. Since

$$\int_0^D \hat{\rho}(x)dx = \int_0^L \hat{\rho}(x)dx - \int_D^L \hat{\rho}(x)dx,$$

we have for $0 \leq x < D$,

$$\hat{\rho}(x) = \frac{1-V_0}{D}.$$

Similarly we find the probability in the other subintervals.

(iii) The entropy of the quasi-implied pdf $\hat{\rho}$ is

$$\begin{aligned} H(\hat{\rho}) &= -\int_0^L \hat{\rho}(x) \log \hat{\rho}(x) dx \\ &= -\int_0^D \frac{1-V_0}{D} \log \frac{1-V_0}{D} dx - \sum_{i=0}^{n-1} \int_{D+K_i}^{D+K_{i+1}} \frac{V_i - V_{i+1}}{K_{i+1}-K_i} \log \frac{V_i - V_{i+1}}{K_{i+1}-K_i} dx \\ &- \int_{D+K_n}^{L} \frac{V_n}{L-D-K_n} \log \frac{V_n}{L-D-K_n} dx. \end{aligned}$$

This leads to

$$H(\hat{\rho}) = -(1-V_0) \log \frac{1-V_0}{D} - V_n \log \frac{V_n}{L-D-K_n}.$$

The first derivative of $H(\hat{\rho})$ with respect to D is

$$\frac{\partial}{\partial D} H(\hat{\rho}) = \frac{1-V_0}{D} + \frac{V_n}{L-D-K_n}.$$

By the first order condition

$$\frac{\partial}{\partial D} H(\hat{\rho}) = 0,$$

we have

$$D = \frac{(1-V_0)(L-K_n)}{1-V_0-V_n}.$$

Hence the quasi-implied PoD is

$$\int_0^D \hat{\rho}(x)dx = \int_0^D \frac{1-V_0}{D}dx = 1-V_0.$$

□

For a put option with payoff $v_i(x) = \mathbf{1}_{x\le D+K_i}$, we obtain the following results.

Corollary 3.2. *(i) The implied pdf for a cash-or-nothing put does not exist.*
(ii) The quasi-implied pdf for a cash-or-nothing put is given by

$$\hat{\rho}(x) = \begin{cases} \frac{V_0}{D}, & 0 \le x < D, \\ \frac{V_{i+1}-V_i}{K_{i+1}-K_i}, & D+K_i \le x < D+K_{i+1} \ \text{for } i = 0,\dots,n-1, \\ \frac{1-V_n}{L-D-K_n}, & D+K_n \le x < L. \end{cases}$$

(iii) The optimal solution D of the problem (5) is

$$\frac{V_0(L-K_n)}{V_n-V_0-1}$$

and the quasi-implied PoD is

$$V_0.$$

In fact, we can directly find the quasi-implied PoD without calculation of the quasi-implied pdf in any cash-or-nothing options. The reason we find the quasi-implied pdf is significant to know the the default threshold as well as the quasi-implied PoD.

4 Concluding Remarks

In a summary, we use Shannon entropy, the quasi-implied pdf and cash-or-nothing call options prices instead of cross-entropy, implied pdf and European call option prices suggested by Capuano [6]. In fact, it merely makes a little difference what to use prior density [5, 6]. Moreover, MED corresponds to the minimum cross-entropy distribution when the prior is the uniform distribution.

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