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Common Fixed Point Theorem for More General Occasionally Noncommuting Mappings

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Abstract. We prove common fixed point theorems for the newly introduced "occasionally C_q-commuting" mappings satisfying generalized nonexpansive condition using subsequential continuity of the mappings involved. Our results are more general than many recent results existing in the literature.

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1. Introduction

Fixed point theorems for pairs of mappings involved additional hypothesis, in addition to contractive condition. First hypothesis is given a space X, containment of range of one mapping into the range of another mapping involved. Second hypothesis is some kind of commutativity condition. Sessa [13] weakened the notion of commutativity as "weakly commuting mappings" which was generalized by Jungck [7] as "compatible mappings" and then to "weakly compatible mappings"[8]. Al-thagafi and Shahzad [1] generalized the notion of weakly compatible mappings as "C_q-commuting mappings" and obtained common fixed point and invariant approximation theorems.

More recently, Al-thagafi and Shahzad [2] introduced two new classes of noncommuting mappings namely "occasionally weakly compatible" and "ultraoccasionally weakly compatible" mappings and established common fixed point and invariant approximation theorems.

On the other hand, Bouhadjera and Thobie [3] introduced a new notion called "subsequential continuous mappings" by weakening the concept of "reciprocal continuous mappings" introduced by Pant [12].

In this paper, weakening the notion of "C_q-commuting mappings", we introduce a new notion called "occasionally C_q-commuting mappings". We obtain common fixed point theorems without using stronger conditions like containment of ranges of the mappings involved and the continuity of the mappings involved. Thus, our results generalizes many recent results existing in the literature.

2. Definitions and Preliminaries

Definition 2.1. *A set M is said to be q-starshaped, if there exists some $q \in M$ such that $kx + (1-k)q \in M$ for all $x \in M$ and $0 \leq k \leq 1$.*

Let T and f be self mappings of a metric space (X, d). The mapping T is called f-contraction mapping, if there exists a real number $0 \leq k < 1$ such that $d(Tx, Ty) \leq kd(fx, fy)$ for all $x, y \in X$.

If $k = 1$, then T is said to be f-nonexpansive. If $f = I$, the identity mapping, then T is called the well known Banach's contraction mapping. If, in addition, $k = 1$, then T is said to be nonexpansive.

A self mapping T of a metric space (X, d) is said to be hemicompact, if any sequence $\{x_n\}$ in X has a convergent subsequence whenever $d(x_n, Tx_n) \to 0$ as $n \to \infty$. A self mapping T of a nonempty subset M of a metric space (X, d) is said to be compact, if $\{Tx_n\}$ has a convergent subsequence $\{Tx_m\}$ in M whenever $\{x_n\}$ is a bounded sequence in M.

Two self mappings f and T of a metric space (X, d) are said to be weakly compatible, if $fTx = Tfx$ whenever $fx = Tx$ for all $x \in X$.

The following are the definitions recently introduced by Al-thagafi and Shahzad [2].

Definition 2.2. [2] *Two self mappings f and T of a metric space (X, d) are said to be occasionally weakly compatible, if $fTx = Tfx$ for some $x \in X$ such that $fx = Tx$.*

The class of occasionally weakly compatible mappings properly contains the class of weakly compatible mappings which is seen from the following example.

Example 2.1. [2] *Let $X = \mathbb{R}$ with the usual norm and $M = [0, \infty)$. Define $f, T : M \to M$ by $fx = 2x$ and $Tx = 2x^2$ for all $x \in M$. Then $C(f, T) = \{0, 1\}$. It is noted that $fT0 = Tf0$ but $fT1 \neq Tf1$. Therefore f and T are occasionally weakly compatible but not weakly compatible.*

Definition 2.3. [2] *Let f and T be two self mappings of a q-starshaped subset M of a normed linear space X and let $T_k x = kTx + (1 - k)q$ for every $x \in M$, $0 \leq k \leq 1$ and for some $q \in M$. Then T and f are said to be ultraoccassionally weakly compatible, if f and T_k are occasionally weakly compatible for every $k \in [0, 1]$.*

The following condition called "reciprocal continuity" was introduced by Pant [12] in 1994.

Definition 2.4. [12] *Two self mappings f and T of a metric space (X, d) are said to be reciprocally continuous if $lim_n fTx_n = ft$ and $lim_n Tfx_n = Tt$ whenever $\{x_n\}$ is a sequence in X such that $lim_n fx_n = im_n Tx_n = t$ for some $t \in X$.*

The following condition called "subsequential continuous" is recently introduced by Bouhadjera and Thobie [3].

Definition 2.5. [3] *Two self mappings f and T of a metric space (X, d) are said to be subsequentially continuous if and only if there exists a sequence $\{x_n\}$ in X such that $lim_n fx_n = lim_n Tx_n = t$ for some $t \in X$ and satisfy $lim_n fTx_n = ft$ and $lim_n Tfx_n = Tt$.*

If f and T are both continuous and reciprocally continuous, then they are obviously subsequentially continuous.

We provide the following example which shows that the pair of mappings which are subsequentially continuous are neither continuous nor reciprocally continuous.

Example 2.2. *Let $X = [0, \infty)$ with the usual metric. Define two self mappings f and T of X by*

$$fx = \begin{cases} 2 + x & if \quad 0 \leq x \leq 2 \\ x \; if & 2 < x < \infty, \end{cases}$$

and

$$Tx = \begin{cases} 2 - x & if \quad 0 \leq x < 2 \\ 2x - 2 \; if & 2 \leq x < \infty, \end{cases}$$

It may be noted that f and T are discontinuous at $x = 2$. Now, consider the sequence $x_n = \frac{1}{n}$ for $n = 1, 2, 3, \ldots$. Then, as $n \to \infty$,

$$\begin{aligned} fx_n &= 2 + x_n \to 2 = t \\ Tx_n &= 2 - x_n \to 2 \; and \\ fTx_n &= f(2 - x_n) = 4 - x_n \to 4 = f(2) \\ Tfx_n &= T(2 + x_n) = 2(2 + x_n) - 2 = 2 + 2x_n \to 2 = T(2), \end{aligned}$$

which shows that f and T are subsequentially continuous. But if we consider another sequence $x_n = 2 + \frac{1}{n}$ for $n = 1, 2, 3, \ldots$, then as $n \to \infty$,

$$\begin{aligned} fx_n &= x_n = (2 + \frac{1}{n}) \to 2 = t \\ Tx_n &= 2x_n - 2 = 2(2 + \frac{1}{n}) - 2 \to 2 \; and \\ fTx_n &= f(2x_n - 2) = 2x_n - 2 = 2(2 + \frac{1}{n}) - 2 \to 2 \neq 4 = f(2). \end{aligned}$$

Lemma 2.1. [10] *Let X be a set, T and f be occasionally weakly compatible self mappings of X. If f and T have a unique point of coincidence, $w := fx = Tx$, then w is the unique common fixed point of f and T.*

3. Main Results

On the same lines of Al-thagafi and Shahzad [2], we introduce the following definition.

Definition 3.1. *Let M be a q-starshaped subset of a normed linear space X. Let f and T be two self mappings of M with $q \in F(f)$. The self mappings f and T of M are said to be occasionally C_q-commuting, if*

$$fTx = Tfx \; for \; some \; x \in C_q(f, T) = \bigcup \{C(f, T_k) : 0 \leq k \leq 1\}$$

where $T_k x = kTx + (1 - k)q$.

The following example shows that C_q-commuting mappings form a subclass of occasionally C_q-commuting mappings.

Example 3.1. *Let $X = \mathbb{R}$ with the usual norm and $M = [0, \infty)$. Define $f, T : M \to M$ by $fx = x$ and $Tx = \frac{x^2}{2}$ for all $x \in M$. Then M is q-starshaped with $q = 0$, $C(f, T) = \{0, 2\}$ and $C_q(f, T) = [0, 2]$. It is noted that $fT0 = Tf0$ but $fT2 \neq Tf2$. Therefore f and T are occasionally C_q-commuting but not C_q-commuting.*

Remark 3.1. *It is clear that if f is affine with respect to q, then occasionally C_q-commuting mappings are ultraoccassionally weakly compatible and the converse is also true. It may be noted that then occasionally C_q-commuting mappings are occasionally weakly compatible but the converse is not true in general.*

The following lemma will be needed in the sequel.

Lemma 3.1. *Let M be a nonempty subset of a metric space (X, d). Suppose that f and T are occasionally weakly compatible self mappings of M, T is a generalized f-contraction mapping satisfying*

(3.1)

$$d(Tx, Ty) \leq k \quad max \left\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2}[d(fx, Ty) + d(fy, Tx)]\right\}$$

for all $x, y \in M$ and some $k \in [0, 1)$, then there exists a unique common fixed point of f and T in M.

Proof. Since f and T are occasionally weakly compatible, there exists $x \in X$, such that $fx = Tx$. We claim that x is the unique point of coincidence. Let $z \neq x \in X$ be such that $fz = Tz$, then

$$\begin{aligned} d(Tx, Tz) &\leq k \quad \max \left\{d(fx, fz), d(fx, Tx), d(fz, Tz), \frac{1}{2}[d(fx, Tz) + d(fz, Tx)]\right\} \\ &\leq k \quad \max \left\{d(Tx, Tz), 0, 0, \frac{1}{2}[d(Tx, Tz) + d(Tz, Tx)]\right\} \\ &\leq kd(Tx, Tz) \end{aligned}$$

a contradiction. Thus $Tx = Tz$ which implies that $fx = fz$. Therefore $w = fx = Tx$ is the unique point of coincidence of f and T. By Lemma 2.1, w is the unique common fixed point of f and T. □

Theorem 3.1. *Let f and T be self mappings of a nonempty q-starshaped subset M of a normed linear space X. Suppose that f and T are occasionally C_q-commuting and subsequentially continuous on M. If f is affine with respect to $q \in F(f)$ and* *the mapping T satisfy*

$$(3.2)\quad \|Tx - Ty\| \leq max\,\{\|fx - fy\|, dist(fx,[Tx,q]), dist(fy,[Ty,q]), \frac{1}{2}[dist(fx,[Ty,q]) + dist(fy,[Tx,q])]\}$$

for all $x, y \in M$, then there exists a unique common fixed point of f and T in M provided one of the following conditions is satisfied.

(i) *$cl[T(M)]$ is compact.*
(ii) *T is hemicompact.*
(iii) *T is compact and M is bounded.*

Proof. Let $\{k_n\}$ be a sequence in $(0,1)$ such that $k_n \to 1$ as $n \to \infty$. Define a mapping T_n by

$$T_n x = k_n T x + (1 - k_n) q \quad \text{for all } x \in M$$

As M is q-starshaped and T is a well defined self mapping of M, $T_n : M \to M$. As f and T are occasionally C_q-commuting, it follows that $fTx = Tfx$ for some $x \in M$ such that $fx = T_n x$. As f is affine with respect to $q \in F(f)$, for the same $x \in M$, we have

$$T_n f x = k_n T f x + (1 - k_n) q = k_n f T x + (1 - k_n) f q = f T_n x$$

Therefore $T_n f x = f T_n x$ for some $x \in M$ such that $fx = T_n x$ which implies that T_n and f are occasionally weakly compatible. Moreover, as T satisfy inequality (3.2), it follows that

$$\begin{aligned}\|T_n x - T_n y\| &= k_n \|Tx - Ty\| \\ &\leq k_n \max\,\{\|fx - fy\|, dist(fx,[Tx,q]), dist(fy,[Ty,q]), \frac{1}{2}[dist(fx,[Ty,q]) + dist(fy,[Tx,q])]\} \\ &\leq k_n \;\max\,\{\|fx - fy\|, \|fx - T_n x\|, \|fy - T_n y\|, \frac{1}{2}[\|fx - T_n y\| + \|fy - T_n x\|]\}\end{aligned}$$

Now, T_n is generalized f-contraction satisfying inequality (3.1). Since (T_n, f) satisfies all the conditions of Lemma(3.1), there exists $x_n \in M$ such that for each n,

$$(3.3)\quad x_n = f x_n = T_n x_n = k_n T x_n + (1 - k_n) q.$$

(i) As $cl[T(M)]$ is compact, $cl[T_n(M)]$ is also compact. Hence the sequence $\{x_n\} = \{T_n x_n\}$ has a subsequence $\{x_m\}$ converging to some $z \in M$. Thus

$$(3.4)\quad lim_m x_m = lim_m f x_m = lim_m T_m x_m = z.$$

As f and T are subsequentially continuous in M, it follows that $lim_{n\to\infty} fx_m = lim_{m\to\infty} Tx_m = z$ and satisfy $lim_{m\to\infty} fTx_m = fz$ and $lim_{m\to\infty} Tfx_m = Tz$. Since $k_m \to 1$ as $m \to \infty$, it follows that

$$T_m fx_m = k_m Tfx_m + (1-k_m)q \to Tz \text{ and } fT_m x_m = k_m fTx_m + (1-k_m)q \to fz \text{ as } m \to \infty.$$

(i.e) f and T_n are also subsequentially continuous in M. Now, from equality (3.4), it follows that

$$T_m fx_m = T_m x_m = fx_m \to z \text{ and } fT_m x_m = fx_m \to z \text{ as } m \to \infty.$$

By the uniqueness of the limit, we have $z = Tz = fz$.

(ii) As $k_n \to 1$ as $n \to \infty$, it follows from equality(3.3) that

$$\|x_n - Tx_n\| = \|fx_n - Tx_n\| \leq (\frac{1}{k_n} - 1)(\|q\| + \|x_n\|) \to 0 \text{ as } n \to \infty.$$

Since T is hemicompact, the sequence $\{x_n\}$ has a subsequence $\{x_m\}$ converging to some $z \in M$. Proceeding as in (i), we obtain $z = Tz = fz$.

(iii) Since T is a compact mapping and the sequence $\{x_n\}$ in M is bounded, $\{Tx_n\}$ has a subsequence $\{Tx_m\}$ converging to some $z \in M$. As $k_m \to 1$ as $m \to \infty$, it follows from equality(3.4) that $lim_m x_m = lim_m T_m x_m = z$. Proceeding as in (i), we obtain $z = Tz = fz$.

□

Remark 3.2. *Theorem 3.1 generalizes many theorems as stronger condition like "$clT[M] \subset f(M)$", is not assumed. Moreover, continuity of the mappings is relaxed to subsequential continuity. Hence Theorem 3.1 generalizes Theorem 2.2 of Hussain and Rhoades* [5]*, Theorem 2.3 and Theorem 2.4 of Al-Thagafi and Shahzad* [2]*.*

The following corollary improves Theorem 2.2 of Al-Thagafi and Shahzad [1].

Corollary 3.1. *Let f and T be self mappings of a nonempty q-starshaped subset M of a normed linear space X. Suppose that f and T are occasionally C_q-commuting and subsequentially continuous on M. If f is affine with respect to $q \in F(f)$ and the mapping T is f-nonexpansive, then there exists a unique common fixed point of f and T in M.*

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