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Kernels and k-Kernels in Orientations of the Path Graph

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Abstract

Let G be a graph. The path graph of G, denoted $T(G)$, is defined as follows:

(a) $V(T(G)$ is the set of paths of G whose length is at least one.

(b) For $h, k \in V(T(G))$, $(h, k) \in E(T(G))$ if and only if they are adjacent as paths in G (i.e. they have only one common endpoint).

In this paper we prove the two following results:

(1) Let D be an orientation of $T(G)$ such that each directed triangle is symmetrical. If each odd directed cycle of D, $\overrightarrow{\mathcal{C}} = (0, 1, \ldots, n-1, 0)$ with $\ell(\overrightarrow{\mathcal{C}}) \geq 5$ has a chord (i, j) such that at least one of the two following properties holds:

(a) $j \notin \{i-2, i+2\}$ or

(b) if $j \in \{i-2, i+2\}$, then there exists another chord of $\overrightarrow{\mathcal{C}}$; (r, s) then D has a kernel.

(2) Let D be an orientation of $T(G)$ such that $\mathrm{Asym}(D)$ is strongly connected and each directed triangle has two symmetrical arcs. If every directed cycle of D, $\overrightarrow{\mathcal{C}} = (0, 1, \ldots, n-1, 0)$ with $\ell(\overrightarrow{\mathcal{C}}) \not\equiv 0 \pmod{k}$

has a chord (i, j) such that at least one of the two following properties holds:

(a) $j \notin \{i-2, i+2\}$ or

(b) if $j \in \{i-2, i+2\}$, then there exists another chord of $\overrightarrow{\mathcal{C}}$; (r, s) with $(r, s) \neq (j, i)$,

then D has a k-kernel, $(k \geq 3)$.

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1 Introduction

For general Graph Theory concepts we refer the reader to [1]. Let G be a graph; $V(G)$ and $E(G)$ will denote the sets of vertices and edges of G respectively. A digraph D is an orientation of G if D is obtained by directing each edge of G in at least one of the two possible directions. If $S \subseteq V(G)$ or $T \subseteq E(G)$, then $G[S]$ and $G[T]$ will denote the subgraphs of G induced by S and T respectively.

Let D be a digraph; $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D respectively. An arc $(u_1, u_2) \in A(D)$ is called asymmetrical (respectively symmetrical) if $(u_2, u_1) \notin A(D)$ (resp. $(u_2, u_1) \in A(D)$). The asymmetrical part of D (resp. symmetrical part of D) which is denoted by $\text{Asym}(D)$ (resp. $\text{Sym}(D)$) is the spanning subdigraph of D whose arcs are the asymmetrical (resp. symmetrical) arcs of D.

If $\mathcal{T} = (0, 1, \ldots, n-1, n)$ is a path of the graph G, then the vertices 0 and n are called the ends of $\mathcal{T}$. The endpoint 0 and path $\mathcal{T}$ are incident with each other, as are n and $\mathcal{T}$. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are distinct paths of G incident with only one common endpoint, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are adjacent paths.

The path graph of G is the graph $T(G) = (V(T(G)), E(T(G)))$ whose vertices set is the set of paths of G whose length is at least one; and for $h, k \in V(T(G))$, $(h, k) \in E(T(G))$ if and only if they are adjacent as paths in G (i.e. they have only one common endpoint). We denote the path $h = (0, 1, \ldots, n-1, n)$ and the vertex $h \in V(T(G))$ by the same symbol.

If $\mathcal{C}$ is a walk of G (resp. a directed walk of D) we will denote by $\ell(\mathcal{C})$ its length.

Along this work all notation will be taken modulo n without more explanation.

A cycle of G (resp. a directed cycle of D) is a sequence of vertices of G (resp. of D), $\mathcal{C} = (0, 1, \ldots, n-1, 0)$, such that $[i, i+1] \in E(G)$ (resp. $(i, i+1) \in A(D)$), for $i \in \{0, 1, \ldots, n-1\}$.

Walks, paths and cycles are partial subgraphs or partial subdigraphs.

Let $\mathcal{C}$ be a cycle of G (resp. a directed cycle of D). For $\{i.j\} \subseteq V(\mathcal{C})$ we denote by $[i, \mathcal{C}, j]$ (resp. by $(i, \mathcal{C}, j)$) the path from i to j, $[i, i+1, i+2, \ldots, j]$ (resp. the directed path $(i, i+1, i+2, \ldots, j)$) contained in $\mathcal{C}$. A chord of $\mathcal{C}$ is an edge (resp. an arc) $[i, j] \in A(G) - A(\mathcal{C})$ (resp. $(i, j) \in A(D) - A(\mathcal{C})$) such that $1 < \ell(i, \mathcal{C}, j) < \ell(\mathcal{C}) - 1$; with $\{i, i+1, \ldots, j\} \subseteq V(\mathcal{C})$. Two vertices joined by an arc of $\mathcal{C}$ are said to be consecutive on $\mathcal{C}$. A pole of the cycle $\mathcal{C}$ is the terminal vertex of a chord (x, y) of $\mathcal{C}$.

By the directed distance $d_D(x, y)$ from the vertex x to vertex y in a digraph D we mean the length of the shortest directed path from x to y in D. We put $d_D(x, y) = \infty$ if there is no directed path from x to y in D.

Let k be a natural number with $k \geq 2$. A set $J \subseteq V(D)$ will be called a k-kernel of the digraph D if:

(1) for $\{x, x'\} \subseteq J$ we have $d_D(x, x') \geq k$ and

(2) for each $y \in (V(D) - J)$ there exists $x \in J$ such that $d_D(y, x) \leq k - 1$.

k-kernels were first defined and studied by M. Kwaśnik in [8]. In [8], M. Kwaśnik proved the following interesting result: Let D be a strongly connected digraph such that every directed cycle of D has length $\equiv 0 (\mathrm{mod}\, k), k \geq 2$, then D has a k-kernel.

For $k = 2$ we have a kernel in the sense of Berge [1]. When every induced subdigraph of D has a kernel, D is said to be kernel-perfect or a KP-digraph.

In 1976 H. Meyniel [3] conjectured: Let D be a digraph; if every odd directed cycle of D possesses two chords, then D is a KP-digraph. In general, the condition that each odd directed cycle has two chords is not sufficient for a digraph to be kernel-perfect. In [4], Galeana-Sánchez constructed for each k a triangle free digraph D_k with no kernel such that every odd directed cycle in D_k has at least k chords. In [7], Galeana-Sánchez and V. Neumann-Lara proved that if every odd directed cycle $\mathcal{C}$ has two chords whose terminal endpoints are consecutive on $\mathcal{C}$, then D is kernel-perfect. Still under some restrictions on the structure of the underlying graph of a digraph D the condition: Each odd directed cycle has two chords is not enough for a digraph to be kernel-perfect. However in [2], O.V. Borodin, A.V. Kostochka and D.R. Woodall proved: Let H be the line graph of a graph G; an orientation D of H is kernel-perfect if and only if each odd directed cycle has a chord and each clique has a kernel.

A feasible extention of the Meyniel's Conjecture for k-kernels $k \geq 2$ would say: Let D be a digraph; if every directed cycle of length$\not\equiv 0 \pmod k$ has two chords, then D has a k-kernel.

In [6], we proved that this assertion is not true for digraphs in general. We proved the following extention of Borodin, Kostochka and Woodall result for k-kernels ($k \geq 3$): Let G be a graph, $L(G)$ its line graph and D an orientation of $L(G)$ such that $\mathrm{Asym}(D)$ is strongly connected and each directed triangle has two symmetrical arcs; if every directed cycle of D, $\overrightarrow{\mathcal{C}} = (0, 1, \ldots, n-1, 0)$ with $\ell(\overrightarrow{\mathcal{C}}) \not\equiv 0 \pmod k$ has a chord (i, j) such that at least one of the two following properties holds:

(1) $j \notin \{i-2, i+2\}$ or

(2) if $j \in \{i-2, i+2\}$, then there exists another chord of $\overrightarrow{\mathcal{C}}$; (r, s) with $(r, s) \neq (j, i)$,

then D has a k-kernel, ($k \geq 3$).

In this paper we under similar conditions as in [6] with extend our results to the path graph: Let G be a graph, $T(G)$ its path graph and D an orientation of $T(G)$ such that each directed triangle is symmetrical; if each odd directed cycle $\overrightarrow{\mathcal{C}} = (0, 1, \ldots, n-1, 0)$ of D whose $\ell(\overrightarrow{\mathcal{C}}) \geq 5$ has a chord (i, j) such that at least one of the two following properties holds:

(1) $j \notin \{i-2, i+2\}$ or

(2) if $j \in \{i-2, i+2\}$, then there exists another chord of $\overrightarrow{\mathcal{C}}$; (r, s)

then D has a kernel.

Let G be a graph, $T(G)$ its path graph and D an orientation of $T(G)$ such that $\mathrm{Asym}(D)$ is strongly connected and each directed triangle has two symmetrical arcs; if every directed cycle of D, $\overrightarrow{\mathcal{C}} = (0, 1, \ldots, n-1, 0)$ with $\ell(\overrightarrow{\mathcal{C}}) \not\equiv 0 \pmod k$ has a chord (i, j) such that at least one of the two following properties holds:

(1) $j \notin \{i-2, i+2\}$ or

(2) if $j \in \{i-2, i+2\}$, then there exists another chord of $\overrightarrow{\mathcal{C}}$; (r, s) with $(r, s) \neq (j, i)$,

then D has a k-kernel, ($k \geq 3$).

As a consequence it is proved the following assertion which is a particular case in which the feasible extention of the Meyniel's Conjecture for $k \geq 3$ holds: Let G be a graph and D an orientation of $T(G)$ such that $\mathrm{Asym}(D)$ is strongly connected and each directed triangle is symmetrical; if every directed cycle of D whose length is $\not\equiv 0 \pmod k$ has two chords, then D has a k-kernel, $k \geq 3$.

The existence of k-kernels of digraphs have been studied by several authors, namely: M. Kwaśnik, A. Wloch and I. Wloch [9], Q. Lu, E. Shan and M. Zhao [10], W. Szumny, A. Wloch and I. Wloch [11], [12], and A. Wloch and I. Wloch [13].

2 Kernels in orientations of the path graph

Lemma 2.1. *Let G be a graph, $T(G)$ its path graph and $\mathcal{C} = (0, 1, \ldots, n-1, 0)$ be a cycle in $T(G)$. If $[i, j] \in E(T(G)) - E(\mathcal{C})$ with $j \notin \{i-2, i+2\}$, then at least one of the following conditions holds:*

(a) $\{[s-1, s+1], [s, t]\} \subseteq E(T(G))$ *with; ($s = i$ and $t \in \{j-1, j+1\}$) or ($s = j$ and $t \in \{i-1, i+1\}$).*

(b) $\{[i-1, i+1], [j-1, j+1]\} \subseteq E(T(G))$.

(c) $T(G)[\{s-1, s, t, t+1\}] \cong K_4$ *with $s \in \{i, i+1\}$, $t \in \{j-1, j\}$.*

Proof: Let G be a graph, $T(G)$ its path graph and $\mathcal{C} = (0, 1, \ldots, n-1, 0)$ be a cycle in $T(G)$, $[i, j] \in E(T(G)) - E(\mathcal{C})$ with $\{i.j\} \subseteq V(\mathcal{C})$ and $j \notin \{i-2, i+2\}$.

Let u and v the terminal vertices of the path i.

We consider several possible cases:

Case 1) The path $i-1$ is incident to u and the path $i+1$ is incident to u.

Clearly in this case we have $[i-1, i+1] \in E(T(G))$.

If the path $j-1$ (resp. $j+1$) is incident to some endpoint of i (u or v), then $[i, j-1] \in E(T(G))$ (resp. $[i, j+1] \in E(T(G))$) and (a) holds with $s = i$ and $t = j-1$ (resp. $s = i$ and $t = j+1$). So we can assume the path $j-1$ is not adjacent to path i and path $j+1$ is not adjacent to i. This there exist w endpoint of j such that $w \notin \{u, v\}$. Since $j-1$ (resp. $j+1$) is not adjacent to i but $j-1$ (and $j+1$) is adjacent to j it follows that $j-1$ (and $j+1$) is incident to w; so $[j-1, j+1] \in E(T(G))$ and (b) holds.

Case 2) The path $i-1$ is incident to v and the path $i+1$ is incident to v.

In this case the proof is exactly as those of Case 1.

Case 3) The path $i-1$ is incident to u and the path $i+1$ is incident to v.

Since the path j is adjacent to path i, we have that j is incident to u or j is incident to v.

First suppose that j is incident to u.

If $j-1$ (resp. $j+1$) is incident to u, then (c) holds with $s = i$ and $t = j-1$ (resp. $s = i$ and $t = j$). So we can assume that both $j-1$ and $j+1$ is incident to the other endpoint of j and then $[j-1, j+1] \in E(T(G))$ and (a) holds with $s = j$ and $t = i-1$. Now suppose that j is incident to v.

When $j-1$ (resp. $j+1$) is incident to v, then (c) holds with $s = i+1$ and $t = j-1$ (resp. $s = i+1$ and $t = j$). When $j-1$ and $j+1$ both is incident to the other endpoint of j we obtain $[j-1, j+1] \in E(T(G))$ and (a) holds with $s = j$ and $t = i+1$.

Case 4) The path $i-1$ is incident to v and the path $i+1$ is incident to u.

Proceed as in Case 3 by interchanging u with v. ■

Lemma 2.2. *Let G be a graph, $T(G)$ its path graph and D an orientation of $T(G)$ such that each directed triangle is symmetrical. If each odd directed cycle $\overrightarrow{\mathcal{C}} = (0, 1, \ldots, n-1, 0)$ of D whose $\ell(\overrightarrow{\mathcal{C}}) \geq 5$ has a chord (i, j) such that at least one of the two following properties holds:*

(1) $j \notin \{i-2, i+2\}$ or

(2) if $j \in \{i-2, i+2\}$, then there exists another chord of $\overrightarrow{\mathcal{C}}$; (r, s)

then each odd directed cycle of D has at least two consecutive poles.

Proof: When $\ell(\overrightarrow{\mathcal{C}}) = 3$, the hypothesis each directed triangle is symmetrical implies that $\overrightarrow{\mathcal{C}}$ has two consecutive poles.

If $\ell(\overrightarrow{\mathcal{C}}) \geq 5$, then we consider the two possible cases:

Case 1) $j \notin \{i-2, i+2\}$

This case implies that $\ell(\overrightarrow{\mathcal{C}}) \geq 7$.

Considering $\mathcal{C}$ and $[i, j] \in E(T(G))$ we have from Lemma 2.1 that at least one of the three properties (a), (b) or (c) holds.

Subcase 1.a) Assume property (a) holds; four possibilities will be analyzed.

1.a.1) $\{[i-1, i+1], [i, j+1]\} \subseteq E(T(G))$. (Considering $s = i$ and $t = j+1$)

If $(i, j+1) \in A(D)$, then j and $j+1$ are two consecutive poles of $\overrightarrow{\mathcal{C}}$.

If $(j+1, i) \in A(D)$, then:

When $(i-1, i+1) \in A(D)$, then i and $i+1$ are two consecutive poles of $\overrightarrow{\mathcal{C}}$.

When $(i+1, i-1) \in A(D)$, then i and $i-1$ are two consecutive poles of $\overrightarrow{\mathcal{C}}$.

1.a.2) $\{[i-1, i+1], [i, j-1]\} \subseteq E(T(G))$. (Considering $s = i$ and $t = j-1$)

Proceed as in (1.a.1) by changing $j+1$ by $j-1$.

1.a.3) $\{[j-1, j+1], [j, i+1]\} \subseteq E(T(G))$. (Considering $s = j$ and $t = i+1$)

1.a.4) $\{[j-1, j+1], [j, i-1]\} \subseteq E(T(G))$. (Considering $s = j$ and $t = i-1$)

Subcase 1.b) Assume that property (b) holds (i.e. $\{[i-1,i+1],[j-1,j+1]\} \subseteq E(T(G))$).

In this cases (1.a.3), (1.a.4), and (1.b) we have $[j-1,j+1] \in E(T(G))$ and since $(i,j) \in A(D)$, then $\overrightarrow{\mathcal{C}}$ has two consecutive poles.

Subcase 1.c) Assume that property (c) holds: Here we have four possibilities.

1.c.1) $T(G)[\{i,i+1,j,j+1\}] \cong K_4$ (Here we are considering $s = i+1$ and $t = j$).

If $(i, j+1) \in A(D)$ or $(i+1, j+1) \in A(D)$, then j and $j+1$ are two consecutive poles of $\overrightarrow{\mathcal{C}}$.

If $(j+1, i) \in A(D)$ or $(j+1, i+1) \in A(D)$, then i and $i+1$ are two consecutive poles of $\overrightarrow{\mathcal{C}}$.

1.c.2) $T(G)[\{i,i+1,j-1,j\}] \cong K_4$ (Considering $s = i+1$ and $t = j-1$).

Proceed as in (1.c.1) by changing $j+1$ by $j-1$.

1.c.3) $T(G)[\{i-1,i,j-1,j\}] \cong K_4$ (Here we are considering $s = i$ and $t = j-1$).

Proceed as in (1.c.2) by changing $i+1$ by $i-1$.

1.c.4) $T(G)[\{i-1,i,j,j+1\}] \cong K_4$ (Considering $s = i$ and $t = j$).

Proceed as in (1.c.1) by changing $i+1$ by $i-1$.

Case 2) $j \in \{i-2, i+2\}$

In this case the hypothesis on Lemma imply that there exists another chord of $\overrightarrow{\mathcal{C}}$, (r,s), and $\ell(\overrightarrow{\mathcal{C}}) \geq 5$

2.1) If $j = i-2$, then $(i, i-2, i-1, i)$ is a symmetrical triangle, that why $\overrightarrow{\mathcal{C}}$ has two consecutive poles.

2.2) If $j = i+2$, then for hypothesis on Lemma imply that there exists another chord of $\overrightarrow{\mathcal{C}}$, (r,s).

It follows from above that $(r,s) \neq (j,i)$ and (r,s) is a short chord. In view of Case 1 we can assume that there exist a, b, with $a \neq b$;

$\{a,b\} \subseteq (0,1,\ldots,n-1,0)$ such that $\{(a-1,a+1),(b-1,b+1)\} \subseteq A(D) - A(\overrightarrow{\mathcal{C}})$, without loss of generality we suppose that $a < b$.

If $a+1 = b$, then b and $b+1$ are two consecutive poles of $\overrightarrow{\mathcal{C}}$.

If $a+1 \neq b$, then we can assume that every diagonal of $\overrightarrow{\mathcal{C}}$ are short and asymmetrical. $(\star)$

Now we consider H subdigraph of D induced by vertices the $\overrightarrow{\mathcal{C}}$.

Let γ be a cycle of minimum length such that $\gamma \subseteq H$.

At least one arc of γ is one diagonal of $\overrightarrow{\mathcal{C}}$.

We will analyze the possible cases:

Case a) $\ell(\overrightarrow{\mathcal{C}})$ is odd.

If $\ell(\overrightarrow{\mathcal{C}}) = 3$, then γ is symmetrical, which implies that a diagonal of $\overrightarrow{\mathcal{C}}$ is symmetrical a contradiction with $(\star)$.

If $\ell(\overrightarrow{\mathcal{C}}) \geq 5$, then by hypothesis, γ has a diagonal (h, l), therefore $(h, l) \cup (h, \gamma, l)$ is a cycle of length shorter than γ within H, a contradiction the choice of γ.

Case b) $\ell(\overrightarrow{\mathcal{C}})$ is even.

For (1) and (2) exists $(x_i, x_i + 1) \in A(\gamma)$ such that is an short chord of $\overrightarrow{\mathcal{C}}$, meanining $(x - i, x_i + 1) = (j, j + 2)$ with $\{j, j + 2\} \subseteq V(\overrightarrow{\mathcal{C}})$.

b.1) If $j + 1 \notin V(\gamma)$, then $(j, j + 1, j + 2) \cup (j + 2, \gamma, j)$ is a cycle of odd length with a short chord $(j, j + 2)$, for hypothesis, it has another diagonal (r, s).

For the choice of γ it follows that $j + 1 \in \{r, s\}$ and in fact by $(\star)$; $(r, s) \in \{(j - 1, j + 1), (j + 1, j + 3)\}$. Therefore $j + 1$ and $j + 2$ (resp. $j + 2$ and $j + 3$) are two consecutive poles.

b.2) If $j + 1 \in V(\gamma)$, then $(j, j + 1) \cup (j + 1, \gamma, j)$ is a directed cycle of length shorther than γ, which contradicts the choice of γ. ■

Theorem 2.3. [7] *If every directed cycle of odd length in D possesses at least two consecutive poles, then D is a kernel-perfect digraph.*

Theorem 2.4. *Let G be a graph, $T(G)$ its path graph and D an orientation of $T(G)$ such that each directed triangle is symmetrical. If each odd directed cycle $\overrightarrow{\mathcal{C}} = (0, 1, \ldots, n - 1, 0)$ of D whose $\ell(\overrightarrow{\mathcal{C}}) \geq 5$ has a chord (i, j) such that at least one of the two following properties holds:*

(1) $j \notin \{i - 2, i + 2\}$ or

(2) if $j \in \{i - 2, i + 2\}$, then there exists another chord of $\overrightarrow{\mathcal{C}}$; (r, s)

then D is a kernel-perfect.

Proof: It follows from Lemma 2.2 each odd directed cycle of D has at least two consecutive poles; then apply 2.3, D is a kernel-perfect. ■

3 k-kernels in orientations of the path graph

Lemma 3.1. *Let G be a graph, $T(G)$ its path graph and $\mathcal{C} = (0, 1, \ldots, n-1, 0)$ be a cycle in $T(G)$. If there exists i, $0 \leq i \leq n-1$ such that $\{[i-1, i+1], [i, i+2]\} \subseteq E(T(G))$, then*

$$\{[i - 1, i + 2], [i, i + 3], [i + 1, i + 3], [i - 2, i], [i - 2, i + 1]\} \cap E(T(G)) \neq \emptyset\,.$$

Proof: Let G be a graph, $T(G)$ its path graph, $\mathcal{C} = (0, 1, \ldots, n-1, 0)$ be a cycle in $T(G)$, the path i, $0 \le i \le n-1$ such that $\{[i-1, i+1], [i, i+2]\} \subseteq E(T(G))$.

Let u and v the terminal vertices of the path i.

We will consider the following possible cases:

Case 1 The path $i-1$ is incident to u and the path $i+1$ is incident to u.

Let z be the endpoint of $i+1$ different from u. Since $[i, i+2] \in E(T(G))$ we have that $i+2$ is incident to u or $i+2$ is incident to v. When $i+2$ is incident to u we obtain $[i-1, i+2] \in E(T(G))$. When $i+2$ is incident to v, the other endpoint of $i+2$ is z. If $i+3$ is incident to v we have $[i, i+3] \in E(T(G))$ and if $i+3$ is incident to z we obtain $[i+1, i+3] \in E(T(G))$.

Case 2 The path $i-1$ is incident to v and the path $i+1$ is incident to v.

This case follows as Case 1 by interchanging u with v.

Case 3 The path $i-1$ is incident to u and the path $i+1$ is incident to v.

Since $[i-1, i+1] \in E(T(G))$ and the path $i+1$ is not incident to u, exist z different from u such that is endpoint the $i-1$ and $i+1$. When $i-2$ is incident to u we obtain $[i-2, i] \in E(T(G))$, and when $i-2$ is incident to z we have $[i-2, i+1] \in E(T(G))$.

Case 4 The path $i-1$ is incident to v and the path $i+1$ is incident to u.

This case follows as Case 3 by interchanging u with v. ■

We say [6] that a graph H satisfies the property $\mathcal{C}^*$ if and only if for each cycle $\mathcal{C} = (0, 1, \ldots, n-1, 0)$ the two following properties hold:

(1) If $[i, j] \in E(H) - E(\mathcal{C})$ with $j \notin \{i-2, i+2\}$, then at least one of the following conditions holds:

(1.a) $\{[s-1, s+1], [s, t]\} \subseteq E(H)$ with; ($s = i$ and $t \in \{j-1, j+1\}$) or ($s = j$ and $t \in \{i-1, i+1\}$).

(1.b) $\{[i-1, i+1], [j-1, j+1]\} \subseteq E(H)$.

(1.c) $H[\{s-1, s, t, t+1\}] \cong K_4$ with $s \in \{i, i+1\}$, $t \in \{j-1, j\}$.

(2) If there exists i, $0 \le i \le n-1$ such that $\{[i-1, i+1], [i, i+2]\} \subseteq E(H)$, then

$$\{[i-1, i+2], [i, i+3], [i+1, i+3], [i-2, i], [i-2, i+1]\} \cap E(H) \neq \emptyset\,.$$

Lemma 3.2. [6] *Let H be a graph satisfying the property $\mathcal{C}^*$, and D an orientation of H such that each directed triangle has two symmetrical arcs. If every directed cycle of D, $\overrightarrow{\mathcal{C}} = (0, 1, \ldots, n-1, 0)$ with $\ell(\overrightarrow{\mathcal{C}}) \not\equiv 0 \pmod{k}$ has a chord (i, j) such that at least one of the two following properties holds:*

(i) $j \notin \{i-2, i+2\}$ *or*

(ii) *if* $j \in \{i-2, i+2\}$, *then there exists another chord of* $\overrightarrow{\mathcal{C}}$; (r,s) *with* $(r,s) \neq (j,i)$,

then every directed cycle of D, $\overrightarrow{\mathcal{C}}$ *with* $\ell(\overrightarrow{\mathcal{C}}) \not\equiv 0 \pmod k$ *has two symmetrical arcs,* $(k \geq 3)$.

Lemma 3.3. [6] *Let* H *be a graph satisfying the property* $\mathcal{C}^*$, *and* D *be an orientation of* H *such that each directed triangle is symmetrical. If each directed cycle of* D *whose length is* $\not\equiv 0 \pmod k$ *has two chords, then each directed cycle of* D *whose length is* $\not\equiv 0 \pmod k$ *has two symmetrical arcs,* $(k \geq 3)$.

Lemma 3.4. *Let* G *be a graph,* $T(G)$ *its path graph and* D *be an orientation of* $T(G)$ *such that each directed triangle is symmetrical. If each directed cycle of* D *whose length is* $\not\equiv 0 \pmod k$ *has two chords, then each directed cycle of* D *whose length is* $\not\equiv 0 \pmod k$ *has two symmetrical arcs,* $(k \geq 3)$.

Proof: It follows from Lemmas 2.1 and 3.1 that $T(G)$ satisfy property $\mathcal{C}^*$, and then apply Lemma 3.3. ∎

Theorem 3.5. [5] *Let* D *be a digraph such that* $\mathrm{Asym}(D)$ *is strongly connected. If every directed cycle of length* $\not\equiv 0 \pmod k$ *has at least two symmetrical arcs then* D *has a* k*-kernel,* $(k \geq 2)$.

Theorem 3.6. *Let* G *be a graph,* $T(G)$ *its path graph and* D *be an orientation of* $T(G)$ *such that* $\mathrm{Asym}(D)$ *is strongly connected and each directed triangle has two symmetrical arcs. If every directed cycle of* D, $\overrightarrow{\mathcal{C}} = (o, 1, \ldots, n-1, 0)$ *with* $\ell(\overrightarrow{\mathcal{C}}) \not\equiv 0 \pmod k$ *has a chord* (i,j) *such that at least one of the following properties holds.*

(i) $j \notin \{i-2, i+2\}$, *or*

(ii) *if* $j \in \{i-2, i+2\}$, *then there exists another chord of* $\overrightarrow{\mathcal{C}}$, (r,s) *with* $(r,s) \neq (j,i)$,

then D *has a* k*-kernel,* $(k \geq 3)$.

Proof: It follows from Lemmas 2.1 and 3.1 that $T(G)$ satisfy property $\mathcal{C}^*$; then apply Lemma 3.2 and Theorem 3.5. ∎

Theorem 3.7. *Let* G *be a graph,* $T(G)$ *its path graph and* D *be an orientation of* $T(G)$ *such that* $\mathrm{Asym}(D)$ *is strongly connected and each directed triangle is symmetrical. If every directed cycle of* D *whose length is* $\not\equiv 0 \pmod k$ *has two chords, then* D *has a* k*-kernel* $(k \geq 3)$.

Proof: It follows from Lemma 3.3 and Theorem 3.6 (as $T(G)$ satisfies the property $\mathcal{C}^*$). ■

Clearly Theorem 3.7 is a particular case in which the feasible extention of Meyniel's Conjecture ennounced in the introduction holds.

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