Int. J. Contemp. Math. Sciences, Vol. 5, 2010, no. 8, 403 - 417

Fourier Transform and Invariant Differential Operators on the Group G_4

K. El-Hussein

Department of Mathematics, Faculty of Science
Al-Jouf University, Saudi Arabia
khali-kh@yahoo.com

Abstract. Let H be the 3−dimensional Heisenberg group and $\mathbb{R}_+^*$, be the multiplicative group of all positive real numbers. Let $G_4 = H \rtimes_\rho \mathbb{R}_+^*$ be the nilpotent Lie group, which is the semi-direct product of H by $\mathbb{R}_+^*$ and Let $\mathcal{U}$ be the complexified universal enveloping algebra of the real Lie algebra $\underline{g}$ of G_4. In this paper the Fourier transform on G_4 is discussed for generalizing the methods in [3] and [1] to prove the existence of a tempered fundamental solution of the invariant differential operator on G_4. Out of these theorem a global solvability of the Lewy operator has been obtained.

Mathematics Subject Classification: 43A30, 35D05

Keywords: Group G_4, Heisenberg Group, Fourier Transform, Invariant Differential Operators, Lewy Operator

1 Results and Introduction.

1.1. Let G_4 be the real nilpotent Lie group of dimension 4, which consists of all matrices of the form:

$$\begin{pmatrix} 1 & x & z \\ 0 & t & y \\ 0 & 0 & 1 \end{pmatrix} \tag{1.1}$$

where $x \in \mathbb{R}$, $y \in \mathbb{R}$, $z \in \mathbb{R}$, and $t \in \mathbb{R}_+^*$. It is shown that the group G_4 contains the Heisenberg H as normal sub-group consisting of all matrices as follows

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2}$$

where $x \in \mathbb{R}$, $y \in \mathbb{R}$, $z \in \mathbb{R}$. Then by[13, $P.238 - 240$], G_4 can be identified with the group $H \underset{\rho_1}{\rtimes} \mathbb{R}_+^*$ semi−direct product of H by$\mathbb{R}_+^*$, via the group homomorphism $\rho_1 : \mathbb{R}_+^* \to Aut(H)$, which is defined by

$$\rho_1(t\ )(z,\ y,\ x\ ) = (z,\ ty,\ t^{\ -1}x\ ) \tag{1.3}$$

for any $x \in \mathbb{R}$, $y \in \mathbb{R}$, $z \in \mathbb{R}$ and $t \ \epsilon\ \mathbb{R}_+^*$, where $\mathbb{R}_+^* = \{x \in \mathbb{R}\ ;\ x \rangle 0\}$ is the multiplicative group of all positive real numbers and $Aut(H)$ is the group of all automorphisms of H. By [13, $P.238 - 240$] and [3, $601 - 612$] the Heisenberg group can be identified with the group $\mathbb{R}^2 \underset{\rho_2}{\rtimes} \mathbb{R}$ via the group homomorphism $\rho_2 : \mathbb{R} \to Aut(\mathbb{R}^2)$, which is defined by

$$\rho_2(x)(z, y) = (z + xy, y) \tag{1.4}$$

Hence the group G_4 can be identified with the group $\mathbb{R}^2 \underset{\rho_2}{\rtimes} \mathbb{R} \underset{\rho_1}{\rtimes} \mathbb{R}_+^*$ of the successive semi−direct product $\mathbb{R}^2, \mathbb{R}$ and $\mathbb{R}_+^*$, where the multiplication of two elements $X = (z, y,\ x;\ t)$ and$Y = (z', y',\ x',\ t')$ in G_4 is given by

$$\begin{aligned} X \cdot Y &= (z, y;\ x;\ t)(z', y',\ x',\ t') \\ &= ((z, y,\ x)(z', ty',\ t^{-1}x'),\ t\ t') \\ &= (z + z' + x\ ty', y + ty',\ x + t^{\ -1}x',\ tt') \end{aligned} \tag{1.5}$$

and the inverse of an element $X \in G_4$ is

$$\begin{aligned} X^{-1} &= (z, y, x, t)^{-1} \\ &= ((\rho_1(t^{-1})((z,\ y,\ x)^{-1}), t^{-1}) \\ &= ((\rho_1(t^{-1})((\rho_2(-x)(-z,\ \ -y)),\ \ -x)), t^{-1}\ ) \\ &= ((\rho_1(t^{-1})(-z + xy,\ \ -y,\ \ -x)), t^{-1}\ ) \\ &= (\rho_2(-tx)(-z,\ \ -t^{-1}y),\ \ -tx,\ t^{-1}) \\ &= (-z + xy,\ \ -t^{-1}y,\ \ -tx,\ t^{-1}) \end{aligned}$$

1.2. Let $C^\infty(G_4)$, $\mathcal{D}(G_4)$, $\mathcal{D}'(G_4)$, $\mathcal{E}'(G_4)$ be the space of $C^\infty-$ functions, $C^\infty-$functions with compact support, distributions and distributions with compact support on G_4. Let $\mathcal{U}$ be the complexified universal enveloping algebra of the real Lie algebra $\underline{g}$ of G_4; which is canonically isomorphic onto the algebra of all distributions on G_4 supported by $\{0\}$, where $0 = (0, 0, 0, 1)$ is the identity element of G_4. For any $u \in \mathcal{U}$ one can define a differential operator P_u on G_4 as follows:

$$\begin{aligned} P_u f(X) &= u * f(X) \\ &= \int_{G_4} f(Y^{-1}X)u(Y)dY \end{aligned} \tag{1.6}$$

for any $f \in C^\infty(G_4)$, where $X = (z,\ y,\ x,\ t)$, $Y = ((z',\ y',\ x',\ t')$, $dY = dz'dy'dx'\frac{dt'}{t'}$ is the Haar measure on G_4 and $*$ denotes the convolution product on G_4. The mapping $u \to P_u$ is an algebra isomorphism of $\mathcal{U}$ onto the algebra of all invariant differential operators on G_4

1.3. Let $B = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_+^*$ be the group of the direct product of $\mathbb{R}^2$, $\mathbb{R}$ and $\mathbb{R}_+^*$, and let $S(B)$ be the symmetric algebra over B. Then there exists a unique linear bijection

$$\lambda :\ S(B) \longrightarrow \mathcal{U}$$

For every $u \in S(B)$, we can associate a differential operator with constant coefficients Q_u on B as follows

$$\begin{aligned} Q_u f(X) &= u *_c f(X) \\ &= f *_c u(X) \\ &= \int_B f(X-Y)u(Y)dY \end{aligned} \tag{1.7}$$

for any $f \in C^\infty(B)$, $X \in B, Y \in B$. where $*_c$ signify the convolution product on the commutative group B and $dY = dz'dy'dx'\frac{dt'}{t'}$ is the Haar measure on B. The mapping $u \mapsto Q_u$ is an algebra isomorphism of $S(B)$ onto the algebra of all invariant differential operators on B, which are nothing but the algebra of all differential operators with constant coefficients on B. For more details see[5, 9]. In this paper we will prove the following results

I- Fourier Transform and Plancherel Formula, theorem 2.1

II-Existence theorem of fundamental solution, see theorem 3.1

III-Solvability of the Lewy operator corollary 4.1 and theorem 4.2

2 Parseval - Plancheral formulas on G_4.

The Schwartz space $\mathcal{S}(G_4)(resp.\mathcal{S}'(G_4))$ of G_4 can be considered as the Schwartz spaces $\mathcal{S}(B) = \mathcal{S}(\mathbb{R}^3 \times \mathbb{R}_+^*)(resp.\mathcal{S}'(B))$ of the direct product of the real vector group $\mathbb{R}^3$ by $\mathbb{R}_+^*$. The actions $\rho_1$of the group $\mathbb{R}_+^*$ on $\mathbb{R}^3$ and ρ_2 of the group $\mathbb{R}$ on $\mathbb{R}^2$ define a natural actions ρ_1 on the dual group $(\mathbb{R}^3)^*$ of the group $\mathbb{R}^3, ((\mathbb{R}^3)^* \simeq \mathbb{R}^3)$ and ρ_2 on the dual group $(\mathbb{R}^2)^*$ of the group $\mathbb{R}^2, ((\mathbb{R}^2)^* \simeq \mathbb{R}^2)$, which are given by :

$$\rho_1(t\ )(\xi_1, \xi_2, \xi_3) = (\xi_1, t\xi_2, t^{-1}\xi_3) \tag{2.1}$$

and

$$\rho_2(x)(\xi_1, \xi_2) = (\xi_1, \xi_2 + x\xi_1) \tag{2.2}$$

So

$$\begin{aligned} \rho_2(x)\rho_1(t\ )\xi &= \rho_2(x)\rho_1(t\ )(\xi_1, \xi_2, \xi_3) \\ &= \rho_2(x)(\xi_1, t\xi_2, t^{-1}\xi_3) \\ &= (\xi_1, t\xi_2 + x\xi_1, t^{-1}\xi_3) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \rho_1(t\ )\rho_2(x)\xi &= \rho_1(t\ )(\rho_2(x)(\xi_1, \xi_2), \xi_3) \\ &= \rho_1(t\ )(\xi_1, \xi_2 + x\xi_1, \xi_3) \\ &= (\xi_1, t\xi_2 + tx\xi_1, t^{-1}\xi_3) \end{aligned} \tag{2.4}$$

for any $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, $x \in \mathbb{R}$ and $t \in \mathbb{R}_+^*$

Definition 2.1. *For every $f \in S(G_4)$, one can define its Fourier transform $\mathcal{F}f$ by*:

$$\mathcal{F}f(\xi, \lambda) = \int\limits_{G_4} f(X,t)\ e^{-i\langle \xi, X\rangle}\ t^{-i\lambda}\frac{dt}{t}dX \tag{2.5}$$

where $\langle \xi, X\rangle = z\xi_1+y\xi_2+x\xi_3$, $X = (z, y, x) \in \mathbb{R}^3$, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, $t \in \mathbb{R}_+^*$, $\lambda \in \mathbb{R}$ *and* $dXdt = dz\ dy\ dx\frac{d}{dt}$ is *the Lebesgue measure on* G_4. *It is clear that the function* $\mathcal{F}f \in \mathcal{S}(G_4)$, *and the mapping* $f \mapsto \mathcal{F}f$ *is an isomorphism from the topological vector space* $\mathcal{S}(G_4)$ *onto itself.*

Theorem 2.1. *The Fourier transform* $\mathcal{F}$ *satisfies :*

$$\overset{\vee}{u} * f(0) = \int\limits_{\mathbb{R}^4} \mathcal{F}f(\xi,\lambda)\ \overline{\mathcal{F}}u(\xi,\lambda)d\lambda d\xi \tag{2.6}$$

for every $f \in S(G_4)$ *and* $u \in \mathcal{E}'(G_4)$, *where* $\overset{\vee}{u}(X,t) = u(X^{-1},t^{-1})$, $\xi = (\xi_1, \xi_2, \xi_3)$, $d\xi d\lambda = d\xi_1 d\xi_2 d\xi_3 d\lambda$, *is the Lebesgue measure on* $\mathbb{R}^4$, $*$ *denotes the convolution product on* G_4 *and* $\overline{\mathcal{F}}$ *is the inverse of the Fourier transform.*

Proof **:** By the classical Fourier transform, we have:

$$\begin{aligned}
\overset{\vee}{u} * f(0) &= \int\limits_{\mathbb{R}^4} \mathcal{F}(\overset{\vee}{u} * f)(\xi,\lambda)d\xi d\lambda \\
&= \int\limits_{\mathbb{R}^4}\int\limits_{G_4} \overset{\vee}{u} * f(X,s')\ e^{-i\langle\xi,X\rangle}\ s'^{-i\lambda}dX\ \frac{ds'}{s'}d\xi d\lambda \\
&= \int\limits_{\mathbb{R}^4}\int\limits_{G_4}\int\limits_{G_4} f((Y,t')(X,s'))u(Y,t')\ e^{-i\langle\xi,X\rangle}\ s'^{-i\lambda}dY\ dX\ \frac{ds'}{s'}\frac{dt'}{t'}d\xi d\lambda.
\end{aligned}$$

By change of variable$(Y,t')(X,s') = (X',t)$ with $(X',t) = (z,y,x,t)$, we get

$$\begin{aligned}
(X,s') &= (Y,t')^{-1}(X',t) = (z',y',x',t')^{-1}(z,y;x;t) \\
&= ((\rho_1(-t')(\rho_2(-x')(-z',-y') + (z,y)), x - x'), t'^{-1}t)
\end{aligned}$$

and

$$\begin{aligned}
& -i\left\langle(\xi,\lambda),(X,s')\right\rangle \\
=\ & -i\left\langle(\xi,\lambda),(Y,t')^{-1}(X,s')\right\rangle \\
=\ & -i\langle(\xi,\lambda),(z',y',x',t')^{-1}(z,y;x;t)\rangle \\
=\ & -i\langle(\xi_1,\xi_2,\xi_3,\lambda),((\rho_1(t'^{-1})(\rho_2(-x')(-z',-y') + (z,y)), x - x'), t'^{-1}t)\rangle \\
=\ & -i\langle(((\rho_2(-x')(\rho_1(t'^{-1})(\xi_1,\xi_2),\xi_3),\lambda),(z - z', y - y', x - x', t'^{-1}t)\rangle \\
=\ & -i\left\langle(\xi_1, t'^{-1}\xi_2 - x'\xi_1, t'\xi_3,\lambda),(z - z', y - y', x - x', t'^{-1}t)\right\rangle
\end{aligned}$$

(2.7)

So, we obtain

$$
\begin{aligned}
& e^{-i\left\langle \xi,(\rho_1(t'^{-1})(z',y',x')^{-1}(z,y;x))\right\rangle}\ (t'^{-1}t)^{-i\lambda} \\
= \ & e^{-i\langle((\xi_1,\xi_2)),\xi_3),(\rho_1(t'^{-1})((\rho_2(-x')((-z',-y')+(z,y)))),\ x-x')}\ (t'^{-1}t)^{-i\lambda} \\
= \ & e^{-i\langle(\rho_2(-x)(\rho_1(t'^{-1})(\xi_1,\xi_2,\xi_3))),((-z',-y')+(z,y),\ x-x')}\ (t'^{-1}t)^{-i\lambda} \\
= \ & e^{-i\left\langle(\xi_1,t'^{-1}\xi_2-x'\xi_1,t'\xi_3),(z-z',\ y-y',x-x')\right\rangle}(t'^{-1}t)^{-i\lambda}
\end{aligned}
$$

By the invariance of the Lebesgue measures, $d\xi_2 d\xi_3$ and $d\xi_2$, we obtain

$$
\begin{aligned}
& \overset{\vee}{u} * f(0) \\
= & \int\limits_{G_4}\int\limits_{G_4}\int\limits_{\mathbb{R}^4} f(X,t)e^{-i\left\langle(\xi_1,t'^{-1}\xi_2-x'\xi_1,t'\xi_3),(z-z',y-y';x-x')\right\rangle}(t'^{-1}t)^{-i\lambda}u(Y,t')dY\ \frac{dt'}{t'}dX\frac{dt}{t}\ d\xi d\lambda \\
= & \int\limits_{G_4}\int\limits_{G_4}\int\limits_{\mathbb{R}^4} f(X,t)e^{-i\langle(\xi_1,\xi_2,\xi_3),(z,y;x)\rangle}t^{-i\lambda}dX\frac{dt}{t}u(Y,t')e^{-i\langle(\xi_1,\xi_2,\xi_3),(-z',-y';-x')\rangle}t'^{i\lambda}dY\frac{dt'}{t'}d\xi d\lambda \\
= & \int\limits_{G_4}\int\limits_{G_4}\int\limits_{\mathbb{R}^4} f(X,t)e^{-i\langle\xi,X\rangle}\ t^{-i\lambda}dX\frac{dt}{t}u(Y,t')e^{i\langle\xi,Y\rangle}\ t'^{i\lambda}dY\ \frac{dt'}{t'}d\xi d\lambda \\
= & \int\limits_{\mathbb{R}^4} \mathcal{F}f(\xi,\lambda)\ \overline{\mathcal{F}}u(\xi,\lambda)d\lambda d\xi
\end{aligned}
$$

where $0=(0,0,0,1)$ is the identity of G_4. The theorem is proved.□

Corollary 2.1. *(i) In Theorem 2.1, if we take* $\overset{\vee}{u}=\widetilde{f}$, *we obtain*

$$
\begin{aligned}
\widetilde{f}*f(0) &= \int\limits_{G_4} |f(X,t)|^2\,\frac{dt}{t}dX \\
&= \int\limits_{\mathbb{R}^4} |\mathcal{F}f(\xi.\lambda)|^2\,d\lambda d\xi
\end{aligned}
\tag{2.8}
$$

where $\widetilde{f}(X,t)=\overline{f(X^{-1},t^{-1})}$, *which is the Plancheral's formula on* G_4.
(ii) If we take $u=\overline{g}\in S(G_4)$, *we find*

$$\int_{G_4} f(X,t)\,\overline{g}(X,t)\frac{dt}{t}dX = \int_{\mathbb{R}^4} \mathcal{F}f(\xi,\lambda)\,\overline{\mathcal{F}g(\xi,\lambda)}d\lambda d\xi \tag{2.9}$$

which is the Parseval formula on G_4.

(iii) The Fourier transform can be extended to an isometry of $L^2(G_4)$.

3 Extension Group and Fundamental Solution.

Let $L = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^* \times \mathbb{R}_+^*$ be the group with law:

$$\begin{aligned} XY &= (z,y;x,r;t,s)(z',y';x',r',t',s') \\ &= ((z,y;x,r)(\rho_1(s)(z',y';x',r')),tt',ss') \\ &= (((z,y;x,r)(z',sy';x',s^{-1}r')),tt',ss') \\ &= (((z,y;x)(\rho_2(r)(z',sy')),x'),s^{-1}r'+r,tt',ss') \\ &= (((z,y;x)+(z'+rsy',sy'),x'),s^{-1}r'+r,tt',ss') \\ &= (z+z'+rsy',y+sy';x+x',s^{-1}r'+r,tt',ss') \end{aligned} \tag{3.1}$$

for all $X = (z,y,x,r,t,s) \in L$ and $Y = (z',y',x',r',t',s') \in L$. In this case the group G_4 can be identified with the closed sub$-$group $\mathbb{R}^2 \times \{0\} \times \mathbb{R} \times \{1\} \times \mathbb{R}_+^*$ of L and B with the subgroup $\mathbb{R}^2 \times \mathbb{R} \times \{0\} \times \mathbb{R}_+^* \times \{1\}$ of L. The group L can be called the extension group of the both groups G_4 and B

Definition 3.1 *For every* $f \in C^\infty(L)$, *one can define a function* $\widetilde{f} \in C^\infty(L)$ *as follows:*

$$\begin{aligned} \widetilde{f}(z,y,x,r,t,s) &= f\ ((\rho_1(t)((\rho_2(x)(z,y)),0,r+x),1,st) \\ &= f\ ((\rho_1(t)(z+xy,y,0,r+x)),1,st) \\ &= f\ (z+xy,ty,0,t^{-1}(r+x),1,st) \end{aligned} \tag{3.2}$$

for any $(z,y,x,r,t,s) \in L$.

Remark 3.1. *The function* $\widetilde{f}$ *is invariant in the following sense:*

$$\widetilde{f}((\rho_1(h)(\rho_2(k)(z,y)), x-k, r+k), th^{-1}, sh) = \widetilde{f}(z,y,x,r,t,s) \tag{3.3}$$

for any $(z,y,x,r,t,s) \in L$, $k \in \mathbb{R}$ *and* $h \in \mathbb{R}_+^*$

Definition 3.2. *If* $u \in \mathcal{U}$ *and* $f \in C^\infty(L)$, *we can define the convolution product of* $\overset{\vee}{u}$ *and* f *on* G_4 *by*

$$\begin{aligned}&\overset{\vee}{u} * f(z,y,x,r,t,s)\\ = &\int\limits_{G_4} f\left[(z',y',r',s')^{-1}(z,y,x,r,t,s)\right] \overset{\vee}{u}(z',y',r',s')dz'dy'dr'\frac{ds'}{s'}\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}(a,b,c,d)(z,y;x,r;t,s) &= ((a,b,c)(\rho_1(d)(z,y,x,r)), t, sd)\\ &= ((a,b,c)(z,dy,x,d^{-1}r), t, sd)\\ &= ((a,b)(\rho_2(c)(z,dy)), x, d^{-1}r+c, t, sd)\\ &= (a+z+cdy, b+dy, x, d^{-1}r+c, t, sd)\end{aligned} \tag{3.5}$$

for any $(a,b,c,d) \in G_4$ *and* $(z,y;x,r;t,s) \in L$.

Proposition 3.1. *For every* $f \in \mathcal{D}(L)$ *and* $u \in \mathcal{U}$, *we have*

$$\int\limits_{\mathbb{R}^2} \mathcal{F}(\overset{\vee}{u} * \widehat{f})\ (\xi,\mu,\lambda,\nu)d\mu dv = \mathcal{F}(\widehat{f})\ (\xi,0,\lambda,1)\ \mathcal{F}(\overset{\vee}{u})\ (\xi,\lambda) \tag{3.6}$$

where $F(\overset{\vee}{u}*\widehat{f})\ (\xi,\mu,\lambda,\nu) = \int\limits_{G_4}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}_+^*} (\overset{\vee}{u}*\widehat{f})(X,r,t,s)e^{-i\langle(\xi,\mu),(X,r)\rangle}t^{-i\lambda}s^{-i\nu}dXdr\frac{dt}{t}\frac{ds}{s}d\mu d\nu$, $\xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3$, $X = (z,y,x)$, $dX = dzdydx$, $\langle(\xi,\mu),(X,r)\rangle = z\xi_1 + y\xi_2 + x\xi_3 + r\mu, r \in \mathbb{R}$, $t \in \mathbb{R}_+^*$ *and* $s \in \mathbb{R}_+^*$

Proof: First, we have

$$
\begin{aligned}
& \overset{\vee}{u} * \widehat{f}(z,y,x,r,t,s) \\
= & \int_{G_4} \widehat{f}\left[(z',y',r',s')^{-1}(z,y,x,r,t,s)\right] \overset{\vee}{u}(z',y',r',s') d\, z' dy' dr' \frac{ds'}{s'} \\
= & \int_{G_4} \widehat{f}[(((\rho_1(s'^{-1})(z',y',r')^{-1}), s'^{-1})(z,y,x,r,t,s))] \overset{\vee}{u}(z',y',r',s') dz' dy' dr' \frac{ds'}{s'} \\
= & \int_{G_4} \widehat{f}[((\rho_1(s'^{-1})((z',y',r')^{-1}(z,y,x,r))), t, ss'^{-1})] \overset{\vee}{u}(z',y',r',s') dz' dy' dr' \frac{ds'}{s'} \\
= & \int_{G_4} \widehat{f}[((\rho_1(s'^{-1})(\rho_2(-r')((z',y')^{-1} + (z,y))), x, r-r'), t, ss'^{-1})] \overset{\vee}{u}(z',y',r',s') dz' dy' dr' \frac{ds'}{s'} \\
= & \int_{G_4} \widehat{f}[((\rho_1(s'^{-1})(\rho_2(-r')(z-z', y-y')), x, r-r'), t, ss'^{-1})] \overset{\vee}{u}(z',y',r',s') dz' dy' dr' \frac{ds'}{s'}
\end{aligned}
$$

By the invariance of $\widehat{f}$, we get:

$$
\begin{aligned}
= & \int_{G_4} \widehat{f}\left[(\rho_1(s'^{-1})((z-z', y-y'), x-r', r), t, ss'^{-1})\right] u(z',y',r',s') dz' dy' dr' \frac{ds'}{s'} \\
& \int_{G_4} \widehat{f}\left[(\rho_1(s'^{-1})((z-z', y-y'), x-r', r), t, ss'^{-1})\right] u(z',y',r',s') dz' dy' dr' \frac{ds'}{s'} \\
= & \int_{B} \widehat{f}\left[z-z', y-y', x-r', r, ts'^{-1}, s)\right] u(z',y',r',s') dz' dy' dr' \frac{ds'}{s'} \\
= & \overset{\vee}{u} *_c \widehat{f}(z,y,x,r,t,s)
\end{aligned}
\tag{3.7}
$$

Second, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \mathcal{F}(\overset{\vee}{u} * \widehat{f})(\xi,\mu,\lambda,v)\ d\mu d\nu \\
= &\int_{G_4}\int_{G_4}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+^*} \widehat{f}(X-Y,r,ts'^{-1},s)\overset{\vee}{u}(Y,s')e^{-i\langle(\xi,\mu),(X,r)\rangle}t^{-i\lambda}s^{-i\nu}dY\,dX\,dr\frac{dt}{t}\frac{ds}{s}\frac{ds'}{s'}d\mu d\nu \\
= &\int_{G_4}\int_{G_4}\int_{\mathbb{R}^3}\int_{\mathbb{R}_+^*} \widehat{f}(X,r,t,s)e^{-i\langle(\xi,\mu),(X,r)\rangle}t^{-i\lambda}\overset{\vee}{u}(Y,s')e^{-i\langle\xi,Y\rangle}s'^{-i\lambda}s^{-i\nu}dY\,dX\,dr\frac{dt}{t}\frac{ds}{s}\frac{ds'}{s'}d\mu d\nu \\
= &\ \mathcal{F}(\widehat{f})\ (\xi,0,\lambda,1)\ \mathcal{F}(\overset{\vee}{u})\ (\xi,\lambda)
\end{aligned} \tag{3.8}$$

where $Y = (z', y', r') \in \mathbb{R}^3$, $dY = dz'dy'dr'$, $dX = dzdydx, t \in \mathbb{R}_+^*, s' \in \mathbb{R}_+^*$, $s \in \mathbb{R}_+^*$, $r \in \mathbb{R}$, $\mu \in \mathbb{R}$ and $\nu \in \mathbb{R}$.

Whence the proposition. Now we can state the following result.

Theorem 3.2. *Every invariant differential operator on G_4 which is not identically 0 has a tempered fundamental solution.*

*Proof***:** For each complex number s with positive real part, we can define a distribution T^s on L by:

$$\langle T^s, f\rangle = \int_{\mathbb{R}^6}\left[\left|\mathcal{F}(\overset{\vee}{u})(\xi,\lambda)\right|^2\right]^s \mathcal{F}(\widehat{f})(\xi,\mu,\lambda,\nu)d\xi d\lambda d\mu\ dv$$

for each $f \in \mathcal{S}(L\ )$. By Atiyah theorems[1], the function $s \mapsto T^s$ has a meromorphic continuation in the whole complex plan, which is analytic at $s = 0$ and its value at this point is the Dirac measure on the group L. Now we can define another distribution, $\widehat{T^s}$, as follows.

$$\begin{aligned}
\left\langle \widehat{T^s}, f\right\rangle &= \left\langle T^s, \widehat{f}\right\rangle \\
&= \int_{\mathbb{R}^6}\left[\left|\mathcal{F}(\overset{\vee}{u})(\xi,\lambda)\right|^2\right]^s \mathcal{F}(\widehat{f})(\xi,\mu,\lambda,\nu)d\xi d\lambda d\mu\ dv
\end{aligned} \tag{3.9}$$

for any $f \in \mathcal{S}(L\ )$ and $s \in \mathbb{C}$, with Re $(s) \geq 0$.

Note that the distribution $\widehat{T^s}$ is invariant in the sense (3.3), and we have

$$\begin{aligned}\left\langle \widehat{u * \widetilde{u} *_c T^s}, f \right\rangle &= \left\langle u * \widetilde{u} *_c T^s, \widehat{f} \right\rangle \\ &= \left\langle T^s, \overset{\vee}{\widetilde{u}} *_c \overset{\vee}{u} * \widehat{f} \right\rangle \\ &= \int_{\mathbb{R}^6} \left[\left| \mathcal{F}(\overset{\vee}{u})(\xi, \lambda) \right|^2 \right]^s \mathcal{F}(\overset{\vee}{\widetilde{u}} *_c \overset{\vee}{u} * \widehat{f})(\xi, \mu, \lambda, \nu) d\xi d\lambda d\mu\ dv\end{aligned}$$

where

$$\widetilde{u}(z, y, x, t) = \overline{u(-z, -y, -x, t^{-1})}$$

and

$$\overset{\vee}{\widetilde{u}} *_c f(z, y, x, t) = \int_B f((z - a, y - b, x - c, tr^{-1})\overset{\vee}{\widetilde{u}}((a, b, c, r) dadbdc \frac{dr}{r} \quad (3.10)$$

is the commutative convolution product on G_4. By proposition 3.1, we get:

$$\left\langle \widehat{u * \widetilde{u} *_c T^s}, f \right\rangle = \int_{\mathbb{R}^6} \left[\left| \mathcal{F}(\overset{\vee}{u})(\xi, \lambda) \right|^2 \right]^s \mathcal{F}(\widehat{f})(\xi, \mu, \lambda, \nu) d\xi d\lambda d\mu\ dv$$

hence

$$\widehat{u * \widetilde{u} *_c T^s} = \widehat{T^{s+1}} \quad (3.11)$$

In view of the invariance(3.3), the restriction of the distributions $\widehat{u * \widetilde{u} *_c T^s} = \widehat{T^{s+1}}$ on the sub-group $\mathbb{R}^2 \times \{0\} \times \mathbb{R} \times \{1\} \times \mathbb{R}_+^* \simeq G_4$ are nothing but the distributions

$$u * \widetilde{u} *_c T^s = T^{s+1}.$$

The distribution T^s can be expanded a round $s = -1$ in the form

$$T^s = \sum_{j=-4}^{\infty} \alpha_j (s + 1)^j$$

where each α_j is a distribution on G_4 But $u * \widetilde{u} *_c T^s = T^{s+1}$ can not have a pole at $s = -1$ (since $T^{\,0} = \delta_{G_4}$) and so we must have:

$$\begin{aligned} u * \widetilde{u} *_c \alpha_j &= 0 \quad \text{for} \quad j < 0 \\ u * \widetilde{u} *_c \alpha_0 &= \delta_{G_4} \end{aligned} \tag{3.12}$$

Whence the theorem.

4 Remark on the Lewy Operator and Hormander Condition.

Lewy [10] had proved that if the equation

$$Lf = (-\partial_x - i\partial_y - 2y\partial_z + 2ix\partial_z)f = g \tag{4.1}$$

has a solution $f \in C^1(\mathbb{R}^3)$ for $g \in C^1(\mathbb{R})$, then g is analytic, and by the Hormander necessary condition $[8, P.156]$, the equation 4.1 does not have any distribution solution.

Since then, and in dealing with the non existence of solutions of partial differential operators it was customary during the last fifty years and it still is to day in larger applications, to appeal to the example of the Lewy operator and Hormander condition which guarantees the non existence of (distribution) solutions of the equation 4.1.

Understanding the nature of the kind of these partial differential operators and their invariance on the Heisenberg group requires the admission of solutions.

K. El- Hussein in [3] have proved the local solvability of the similar Lewy operator

$$i\partial_x + \partial_y + iy\partial_z \tag{4.2}$$

Definition4.1. *For every function* $f \in C^\infty(\mathbb{R}^3)$, one can *define a function* $\Lambda(f\ ) \in C^\infty(\mathbb{R}^3)$, *by the following manner*

$$\Lambda(f\ )(z, y\ , x\ ) = f\ (z - 2xy, y\ , -x\ ) \tag{4.3}$$

It is clear that $\Lambda^2 = I$, where I is the identity operator of $C^\infty(\mathbb{R}^3)$.

Theorem 4.1. *For any* $f \in C^{\infty}(\mathbb{R}^3)$*, we get*

$$Pf(z, y, x) = \Lambda Q \Lambda f(z, y, x) \tag{4.4}$$

where $Q = \partial_x - i\partial_y$ and $P = -\partial_x - 2y\partial_z - i\partial_y - 2ix\partial_z$

Proof: In fact if $f \in C^{\infty}(\mathbb{R}^3)$, then we have

$$\begin{aligned} & \Lambda(\partial_x)\Lambda f(z, y, x) \\ = \; & (\partial_x)\Lambda f\ (z - 2xy, y\ , -x\ ) \\ = \; & (\frac{d}{dt})_0 \Lambda f(z - 2xy, y, -x + t) \\ = \; & (\frac{d}{dt})_0 f(z - 2yt, y, x - t) \\ = \; & (-\partial_x - 2y\partial_z) f(z, y, x) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} & \Lambda(-i\partial_y)\Lambda f(z, y, x) \\ = \; & (-i\partial_y)\Lambda f\ (z - 2xy, y\ , -x\ ) \\ = \; & (-i\frac{d}{dt})_0 \Lambda f(z - 2xy, y + t, -x) \\ = \; & (-i\frac{d}{dt})_0 f(z + 2xt, y + t, x) \\ = \; & (-i\partial_y - 2x\partial_z) f(z, y, x) \end{aligned} \tag{4.6}$$

Finally for any $f \in C^{\infty}(\mathbb{R}^3)$, we get

$$\begin{aligned} (-\partial_x - 2y\partial_z - i\partial_y - 2ix\partial_z) f(z, y, x) &= Pf(z, y, x) \\ &= \Lambda Q \Lambda f(z, y, x) \end{aligned}$$

and

$$\begin{aligned} (-\partial_x - 2y\partial_z - i\partial_y + 2ix\partial_z) f(z, y, x) &= Lf(z, y, -x) \\ &= \Lambda Q \Lambda f(z, y, -x) \end{aligned} \tag{4.7}$$

Corollary 4.1. *The Lewy operator* $L = -\partial_x - 2y\partial_z - i\partial_y + 2ix\partial_z$, *verifies the following property*

$$LC^{\infty}(\mathbb{R}^3) = C^{\infty}(\mathbb{R}^3) \tag{4.8}$$

Proof. In fact for any function $g \in C^{\infty}(\mathbb{R}^3)$, there is a function f such that

$$[Pf](z, y, -x) = Lf(z, y, -x) = g(z, y, -x) \tag{4.9}$$

If $\phi \in C_0^{\infty}(\mathbb{R}^3)$, we denote by $\widehat{\phi}$ the function defined by

$$\widehat{\phi}(z, y, x) = \phi(z, y, -x)$$

then the mapping $T \to \widehat{T}$ defined by

$$\langle \widehat{T}(z, y, x), \phi(z, y, x)\rangle = \langle T(z, y, x), \widehat{\phi}(z, y, x)\rangle$$

is a topological isomorphism of $\mathcal{D}'(\mathbb{R}^3)$.

Theorem 4.2. *The Lewy operator* L *has a fundamental solution*

Proof: Let T be a fundamental solution of the operator P, then for any $\phi \in C_0^{\infty}(\mathbb{R}^3)$, we get

$$\begin{aligned} \langle L\widehat{T}(z, y, x), \phi(z, y, x)\rangle &= \langle T(z, y, x), \widehat{{}^tL\phi}(z, y, x)\rangle \\ &= \langle T(z, y, x),{}^t P\phi(z, y, -x)\rangle \end{aligned} \tag{4.10}$$

Changing the variable x by $-x$, we obtain

$$\begin{aligned} & \langle L\widehat{T}(z, y, x), \phi(z, y, x)\rangle \\ = \ & \langle T(z, y, -x), (\partial_x + 2y\partial_z + i\partial_y - 2ix\partial_z)\phi(z, y, x)\rangle \\ = \ & \langle (-\partial_x - 2y\partial_z - i\partial_y + 2ix\partial_z)T(z, y, -x), \phi(z, y, x)\rangle \\ = \ & \langle (-\partial_x - 2y\partial_z - i\partial_y - 2ix\partial_z)T(z, y, x), \phi(z, y, -x)\rangle \\ = \ & \langle \delta_{\mathbb{R}^3}(z, y, x), \phi(z, y, -x)\rangle \\ = \ & \phi(0, 0, 0) \\ = \ & \langle \delta_{\mathbb{R}^3}(z, y, x), \phi(z, y, x)\rangle \end{aligned} \tag{4.11}$$

where ${}^tL$ ($resp.{}^tP$) is the transpose of the operator L ($resp.P$). Then we have

$$L\widehat{T}(z, y, x) = \delta_{\mathbb{R}^3}(z, y, x) \tag{4.12}$$

Consequently if T is a fundamental solution of P, then $\widehat{T}$ is a fundamental solution of L.

References

[1] M. F. Atiyah, *Resolution of Singularities and Division of Distributions*, Comm. on Pure and App. Math, Vol, 23, 145-150, (1970).

[2] K. El- Hussein, *On the Existence Theorem for the Invariant Differential Operators on the Affine Group*, International Journal of Nonlinear Operators Theory and Applications Volume1, No. 1, June 2006, 35-4

[3] K. El- Hussein, *A Fundamental Solution of an Invariant Differential Operator on the Heisenberg Group,* Mathematical Forum, 4, 2009, no. 12, 601 - 612.

[4] K. El- Hussein, *Research Announcements . Unsolved Problems ,* International Mathematical Forum, 4, 2009, no. 12, 597-600.

[5] K. El- Hussein, *Opérateurs Différéntiels Invariants sur les Groupes de Deplacements,* Bull. Sc. Math. 2^e series 113, 1989. p. 89-117.

[6] K. El- Hussein, *Eigendistributions for the Invariant Differential Operators on the Affine Group.* Int. Journal of Math. Analysis, Vol. 3, 2009, no. 9, 419-429.

[7] K. El- Hussein, *R′esolubilit′e Semi-Globale des Op′erateurs Diff′er′entiels Invariants Sur Les Groupe de D′eplacements*, Pacific J. Maths.Vol 142, No 2, 1990.

[8] L. Hormander, *Linear Partial Differential Operators,* Springer-Verlag, Berlin, 1963.

[9] H. Helgason, *Groups and Geometric Analysis*, Academic Press,1984.

[10] H. Lewy, *An Example of a Smooth Linear Partial Differential Equation Without Solution.* Ann. Math. (2) 66, 155-158-(1957).

[11] W. Rudin, *Fourier Analysis on Groups*, Interscience publ, New yourk, 1962.

[12] F. Treves, *Linear Partial Differential Equations with Constant Coefficients,* Gordon and Breach, 1966.

[13] V. S. Varadarajan, *Lie Groups, Lie Algebra and their Representations*, Prentice Hall. Engle-wood Cliffs, New Jersey, 1974.

[14] N.R. Wallach, *Harmonic Analysis on Homogeneous Space*, Marcel Dekker, INC, New York, 1973.

Received: April, 2009