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Tempered Fundamental Solution and Eigendistributions for the Invariant Differential Operators on the Heisenberg Group

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Abstract

It is known that $\chi_\xi(x) = e^{-i<\xi,x>}$ are eigendistributions for any invariant differential operator on $\mathbb{R}^m$. In this paper, we show that if T is an eigendistribution of an invariant differential operator on $\mathbb{R}^m$, then an eigendistribution for an invariant differential operator on the Heisenberg group H^n of dimension $m = 2n+1$ can be obtained . To this end, an existence theorem is put forward. Out of this theorem a tempered fundamental solution of these invariant differential operators has been also obtained.

Mathematics Subject Classification: 43A80

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1 Introduction.

In this paper we use the Fourier transform on the $2n+1-$dimensional Heisenberg group H^n,which is well defined and discussed in [7] to solve the equation $PT = f$, for any distribution f on H^n(*see* theorem 3.1). Moreover if f is a tempered distribution, then T is a tempered distribution (theorem 2.1). These results are obtained by the strong relationship between the invariant differential operators on $\mathbb{R}^{2n+1}$, and the invariant differential operators on H^n. (see proposition 2.1 and proposition 2.2). In addition, the functions which define the Fourier transform on $\mathbb{R}^{2n+1}$, are not only eigenfunctions for the invariant differential operators on $\mathbb{R}^{2n+1}$, but by the exploitation of the properties of the invariance of H^n, these functions will be also eigendistributions for the invariant differential operators on H^n, (see corollary 3.2).

Therefore Let H^n be the real Heisenberg group of dimension $2n+1$, which consists of all matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & I & y \\ 0 & 0 & 1 \end{pmatrix} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}$ and I is the identity matrix of order n.

Let $G = \mathbb{R}^{n+1} \underset{\rho}{\propto} \mathbb{R}^n$ be the group of the semi-direct product of the group $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$, via the group homomorphism $\rho : \mathbb{R}^n \to Aut(\mathbb{R})$, which is defined by

$$\rho(x)(z, y) = (z + xy, y) \tag{1.2}$$

for any $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, $z \in \mathbb{R}$, and $xy = \sum_{i=1}^{n} x_i y_i$. Here, $Aut(\mathbb{R}^{n+1})$ is the group of all automorphism of $\mathbb{R}^{n+1}$. The group H^n can be identified with the group G , see $[12, P.238 - 240]$ or [7].

Let $C^\infty(G)$, $\mathcal{D}(G)$, $\mathcal{D}'(G)$ and $\mathcal{E}'(G)$ be the spaces of C^∞-functions, $C^\infty$with compact supports, distributions, and distributions with compact support on G respectively. Let $\mathcal{U}$ be the complexified universal enveloping algebra of the real Lie algebra $\underline{g}$ of G ; which is canonically isomorphic onto the algebra of all distributions on G supported by $\{0\}$, where 0 is the identity element of G. For any $u \in \mathcal{U}$, a differential operator P_u on G can be defined as follows:

$$\begin{aligned} Puf(X) &= u \star f(X) \\ &= \int_G f(Y^{-1}X) u(Y) dY \end{aligned}$$

for any $f \in C^\infty(G)$,where $dY = dz'dy'dx'$ is the Haar measure on G which is Lebesgue measure on $\mathbb{R}^{2n+1}$, $Y = ((z', y'); x')$, $X = (z, y); x)$, and $\star$ denotes the convolution product on G. The mapping $u \mapsto P_u$ is an algebra isomorphism of $\mathcal{U}$ onto the algebra of all invariant differential opertors on G $[3, 8]$. We denote by $V = \mathbb{R}^{n+1} \times \mathbb{R}^n$ the real vector group which is the direct product of $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$, and we denote by $S(V)$ the symmetric algebra over V. Then there exists a unique linear bijection $\lambda : S(V) \longrightarrow \mathcal{U}$, and the algebra $S(V)$ is isomorphic onto the algebra $D(V)$ of the differential operators with constant coefficients on V.

2 The Division of Distibutions on G.

Let $K = \mathbb{R}^{n+1} \times \mathbb{R}^n \times \mathbb{R}^n$be the group of the mixed product of $\mathbb{R}^{n+1}, \mathbb{R}^n$and $\mathbb{R}^n$, with multiplication

$$\begin{aligned} X \cdot Y &= ((z,y);x,t)((z',y');x',t') \\ &= (((z,y)+t(z',y'));x+x',t+t') \\ &= ((z+z'+ty',y+y');x+x',t+t') \end{aligned}$$

for any $X = ((z,y);x,t) \in K$ and $Y = ((z',y');x',t') \in K$

Definition 2.1: *For every* $f \in \mathcal{D}(K\ )$, *one can define a function* $\widehat{f} \in C^\infty(K)$, *as follows:*

$$\widehat{f}((z,y);x,t) = f(x(z,y);0,x+t) \tag{2.1}$$

for any $z \in \mathbb{R}\ ,\ y \in \mathbb{R}^n, x \in \mathbb{R}^n$ *and* $t \in \mathbb{R}^n$.

Note that the function $\widehat{f}$ is invariant in the following sense :

$$\widehat{f}((k(z,y));x-k,t+k) = \widehat{f}((z,y);x,t) \tag{2.2}$$

for any $z \in \mathbb{R}\ ,\ y \in \mathbb{R}^n,\ x \in \mathbb{R}^n\ ,\ t \in \mathbb{R}^n$ and $k \in \mathbb{R}^n$.

Definition 2.2: *For every* $f \in \mathcal{S}(G)$, *one can define its Fourier transform* $\mathcal{F}f$ *by :*

$$\mathcal{F}f(\xi) = \int\limits_G f(X)\ e^{-i\langle \xi, X\rangle}\ dX \tag{2.3}$$

where $X = ((z,y);x) \in G$, $\xi = ((\eta,\lambda);\mu) \in G$

$$\langle \xi, X\rangle = z\eta + y\lambda + x\mu = z\eta + \sum_{i=1}^n \lambda_i y_i + \sum_{i=1}^n x_i \mu_i$$

and $dX = dz\ dy\ dx$ *the Lebesgue measure on* G. *It is clear that the function* $\mathcal{F}f \in \mathcal{S}(G)$ *and the mapping* $f \mapsto \mathcal{F}f$ is *an isomorphism from the topological vector space* $\mathcal{S}(G)$ *onto it self.*

Now let $u \in \mathcal{U}$, then we can associate two invariant differential operators, the first on G and the second on V. These two operators can be acted on

$C^\infty(K)$ as follows:

$$\begin{aligned}(i)\quad u\star F((z,y);x,t) &= \int_G F((a,b);c)^{-1}((z,y);x,t)u((a,b);c)da\ db\ dc\\ &= \int_G F((-c(z-a,y-b));x,t-c)u((a,b);c)da\ db\ dc\\ (ii)\quad u\star_c F((z,y);x,t) &= \int_V F((z-a,y-b);x-c,t)u((a,b);c)da\ db\ dc\end{aligned} \tag{2.4}$$

for any $F\in C^\infty(K)$ and $((z,y);x,t)\in K$.

Proposition 2.1: *For every* $F\in C_I^\infty(K)$,

$$u\star F((z,y);x,t) = u\star_c F((z,y);x,t) \tag{2.5}$$

for every $((z,y,x,t)\in K$.

According to (2.2), we have

$$F((-c(z-a,y-b));x,t-c) = F((z-a,y-b;x-c,t)$$

for any $F\in C_I^\infty(K)$, $(z,y,x,t)\in K,$and $((a,b);c)\in G$. Thus (2.5) gives the proof of the proposition

The Schwartz space $\mathcal{S}(G)$ of G can be considered as the Schwartz space $\mathcal{S}(\mathbb{R}^{2n+1})$ of the vector group $\mathbb{R}^{2n+1}$. Let $\mathcal{S}'(G)$ be the space of all tempered distributions on G. The action ρ of the group $\mathbb{R}^n$ on $\mathbb{R}^{n+1}$ defines a naturarl action ρ on the dual group $(\mathbb{R}^{n+1})^*$ of the group $\mathbb{R}^{n+1}$ $((\mathbb{R}^{n+1})^*\simeq\mathbb{R}^{n+1})$, which is given by :

$$x(\eta,\lambda) = (\eta,\eta x+\lambda) \tag{2.6}$$

for any $\lambda\in\mathbb{R}^n, x\in\mathbb{R}^n$ and $\eta\in\mathbb{R}$, where :

$$x(\eta,\lambda)=\rho(x)(\eta,\lambda)\ \text{ and }\quad \eta x=\sum_{i=1}^n \eta x_i$$

In order to prove our mai rseultat, we need the following$[see, 8]$

Proposition 3.2: *For every* $f\in\mathcal{S}(G\ )$ *and* $u\in\mathcal{U}$, *we have:*

$$\int_{\mathbb{R}^n} \mathcal{F}(\overset{\vee}{u} * \widehat{f})\ (\xi, v)\ \ dv = \mathcal{F}(\widehat{f})\ (\xi, 0)\ \mathcal{F}(\overset{\vee}{u})\ (\xi) \tag{2.7}$$

where $\xi = (\eta, \lambda, \mu) \in G$. Now we can state the following Theorem.

Theorem 2.1. *Every invariant differential operator on* G *which is not identically* 0 *has a tempered fundamental solution.*

Proof : For each complex number s with positive real part, we can define a distribution T^s on K by:

$$\langle T^s, f\rangle = \int_G \int_{\mathbb{R}^n} \left[\left|\mathcal{F}(\overset{\vee}{u})(\xi)\right|^2\right]^s \mathcal{F}(f)(\xi, v)\ d\xi\ dv$$

for each $f \in \mathcal{S}(K)$. By Atiyah-Bernstein theorems [1, 2], the function $s \mapsto T^s$ has a meromorphic continuation in the whole complex plan, which is analytic at $s = 0$ and its value at this point is the Dirac measure on the group K. Now we can define another distribution, $\widehat{T^s}$, as follows.

$$\begin{aligned} \left\langle \widehat{T^s}, f\right\rangle &= \left\langle T^s, \widehat{f}\right\rangle \\ &= \int_G \int_{\mathbb{R}^n} \left[\left|\mathcal{F}(\overset{\vee}{u})(\xi)\right|^2\right]^s \mathcal{F}(\widehat{f})(\xi, v)\ d\xi\ dv \end{aligned} \tag{2.8}$$

for any $f \in \mathcal{S}(K)$ and $s \in \mathbb{C}$, with $\mathrm{Res}(s) \geq 0$.

Note that the distribution $\widehat{T^s}$ is invariant in sense (2.2) and we have

$$\begin{aligned} \left\langle \widehat{u * \widetilde{u} * c T^s}, f\right\rangle &= \left\langle u * \widetilde{u} * cT^s, \widehat{f}\right\rangle \\ &= \left\langle T^s, \overset{\vee}{\widetilde{u}} * c\overset{\vee}{u} * \widehat{f}\right\rangle \\ &= \int_G \int_{\mathbb{R}^n} \left[\left|\mathcal{F}(\overset{\vee}{u})(\xi)\right|^2\right]^s\ \mathcal{F}(\overset{\vee}{\widetilde{u}} * c\overset{\vee}{u} * \widehat{f})(\xi, v)\ d\xi\ dv \end{aligned}$$

here

$$\widetilde{u}((z, y); x) = \overline{u((-z, -y); -x)}$$

and

$$\overset{\vee}{\widetilde{u}} *_c f = \int_G f((z-a, y-b); x-c)\overset{\vee}{\widetilde{u}}((a,b);c)dadbdc \tag{2.9}$$

is the commutative convolution product on G. By proposition 3.2, we get:

$$\left\langle \widehat{u * \widetilde{u} * c}T^s, f \right\rangle = \int_G \int_{\mathbb{R}^n} \left[\left|\mathcal{F}(\overset{\vee}{u})(\xi)\right|^2\right]^{s+1} \mathcal{F}(\widehat{f})(\xi, v)\ d\xi\ dv$$

hence

$$\widehat{u * \widetilde{u} * c}T^s = \widehat{T^{s+1}} \tag{2.10}$$

In view of invariance (2.2), the restriction of the distributions $\widehat{u * \widetilde{u} * c}T^s = \widehat{T^{s+1}}$ on the sub-group $\mathbb{R}^{n+1} \times \{0\} \underset{\rho}{\propto} \mathbb{R}^n \simeq G$ are nothing but the distributions

$$u * \widetilde{u} * cT^s = T^{s+1}. \tag{2.11}$$

The distribution T^s can be expanded a round $s = -1$ in the form

$$T^s = \sum_{j=-(2n+1)}^{\infty} \alpha_j (s+1)^j$$

where each α_j is a distribution on G. But $u * \widetilde{u} * cT^s = T^{s+1}$ can not have a pole at $s = -1$ (since $T^0 = \delta_G$) and so we must have:

$$\begin{aligned} u * \widetilde{u} * c\alpha_j &= 0 \quad \text{for} \quad j < 0 \\ u * \widetilde{u} * c\alpha_0 &= \delta_G \end{aligned} \tag{2.12}$$

3 An Existence Theorem and Eigendistributions.

In the following, we state and prove an existence theorem, which leads us to deduce the eigendistributions of an element $u \in \mathcal{U}$.

Theorem 3.1: *Let $P_{\breve{u}}$ and $Q_{\breve{u}}$ be the invariant differential operators on G and V respectively, then the following conditions are equivalent:*

$$(i)\;\; Q_{\breve{u}}\mathcal{D}^{'}(V\;) = \mathcal{D}^{'}(V\;).$$

$$(ii)\;\; P_{\breve{u}}\mathcal{D}^{'}(G) = \mathcal{D}^{'}(G). \tag{3.1}$$

Proof : By equation(2.5), we have

$$\begin{aligned} < i(\widetilde{\breve{u}\star_c T}), F > &= < i(\breve{u}\star_c T), \tilde{F} > \\ &= < (\breve{u}\star_c T)\otimes\delta, \tilde{F} > \\ &= < (\breve{u}\star_c (T\otimes\delta), \tilde{F} > \\ &= < T\otimes\delta, u\star_c \tilde{F} > \\ &= < T\otimes\delta, u\star \tilde{F} > \\ &= < iT, u\star \tilde{F} > \\ &= < \breve{u}\star iT, \tilde{F} > \\ &= < \widetilde{\breve{u}\star iT}, F > \end{aligned}$$

for any $F \in \mathcal{D}(K)$and $T\phi \in \mathcal{D}^{'}(V)$. Thus

$$\widetilde{\breve{u}\star iT} = i(\widetilde{\breve{u}\star_c T}) \tag{3.2}$$

Now, as well known in the theory of differential operators with constant coeficients, for every distribution S on V, there is a distribution T on V [11], such that

$$\breve{u}\star_c T = S \tag{3.3}$$

Consequently,

$$\begin{aligned} i(\widetilde{\breve{u}\star_c T}) &= \widetilde{\breve{u}\star iT} = \widetilde{\breve{u}\star (T}\otimes\delta) \\ &= i\widetilde{S} = \widetilde{S\otimes\delta} \end{aligned}$$

By the restriction on G, we get

$$\begin{aligned} (\breve{u}*_c T)\otimes\delta((z,y);0,t) &= \breve{u}\star(T\otimes\delta)((z,y);0,t) \\ &= S\otimes\delta((z,y);0,t) \end{aligned}$$

for any $((z,y),t)\epsilon G$. Let θ be any distribution on $\mathbb{R}^n$, then

$$\begin{aligned}\breve{u}\star(T\otimes\delta)\star_e\theta((z,y);0,t) &= \breve{u}\star(T\otimes\theta)((z,y);0,t)\\ &= S\otimes\theta((z,y);0,t)\end{aligned}$$

where:

$$\breve{u}\star(T\otimes\delta)\star_e\theta((z,y);x,t)=\int_{R^n}\breve{u}\star(T\otimes\delta)((z,y);x,k)\theta(t-k)dk$$

Since $\mathcal{D}'(\mathbb{R}^{n+1})\otimes\mathcal{D}'(\mathbb{R}^n)$ is dense in $\mathcal{D}'(G)$, then for any distribution S' on G there is a distribution E on G such that

$$\breve{u}\star E((z,y);0,t)=S'((z,y);0,t) \tag{3.4}$$

for any $((z,y),t)\in G$. This proves that (i) implies (ii).

In order to prove that (ii) implies(i), consider the group K and the mapping $F\mapsto\tilde{F}$, which is defined by

$$\tilde{F}((z,y);x,t)=F((-x(z,y));0,x+t)$$

for any $F\in C^\infty(K)$ and $((z,y);x,t)\in K$. Then it is easy to show that (ii) implies (i).

Corollary 3.1. *If $T((z,y);x)$ is a fundamental solution for $\breve{u}\in D\ (V)$,* then $iT((z,y);0,t)$ *is also a fundamental solution for* $\breve{u}\in\mathcal{U}$.

Theorem 3.2: *If $T((z,y);x)$ is an eigendistribution of an element $u\in S(V)$, then $iT((z,y);0,t)$ is an eigendistribution of u considered as an invariant differential operator on G.*

Proof : In fact we have:

$$\begin{aligned}\widetilde{\breve{u}\star iT}((z,y);x,t) &= i(\widetilde{\breve{u}\star_c T})((z,y);x,t)\\ &= \widetilde{\breve{u}\star_c iT}((z,y);x,t)\end{aligned}$$

Taking the restriction on G, we get

$$\begin{aligned}\breve{u}\star iT((z,y);0,t) &= i(\breve{u}\star_c T)((z,y);0,t)\\ &= \breve{u}\star_c iT((z,y);0,t)\end{aligned}$$

If T is an eigendistribution of $\breve{u}$, as an invariant differential operator on V, we obtain

$$\widetilde{\breve{u}\star iT} = \widetilde{\lambda iT} = \widetilde{\breve{u}\star_c iT} \tag{3.5}$$

thus, we get

$$\breve{u}\star iT((z,y);0,t) = \lambda iT((z,y);0,t) \tag{3.6}$$

Convolving each side of equation(3.6)by a distribution on $\mathbb{R}^n$ with respect to t, we have

$$\breve{u}\star(T\otimes\theta)((z,y);0,t) = \lambda(T\otimes\theta)((z,y);0,t)$$

equivalently;

$$(T\otimes\theta)\breve{}\star u((z,y);0,t) = \lambda(T\otimes\theta)\breve{}((z,y);0,t)$$

Now, let T_ξ be the distribution on V defined by:

$$\begin{aligned} < T_\xi, f > &= < T_\xi(X), f(X) > \\ &= \int_V f(X)e^{-i<\xi,X>}\,dX \end{aligned} \tag{3.7}$$

for any $f \in \mathcal{D}(V)$, where $X = (z,y);x)\epsilon V, \xi = ((\lambda,\tau);\mu)\epsilon V$, and $< \xi, X >= \lambda z + \sum_{i=1}^n \tau_i y_i + \sum_{i=1}^n \mu_i x_i$, then we have:

Corollary 3.2. *Let $T_\xi((z,y);x)$ be the distributions on V defined by* (3.7), *then the distributions $T_\xi \otimes \chi_\alpha((z,y);0,t)$ on G are eigendistributions for any invariant differential operator on G, where $\chi_\alpha = e^{-i<\alpha,t>}$.*

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