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Generalized Köthe-Toeplitz Duals of Some Difference Sequence Spaces

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Dedicated to Late Prof. B. P. Mishra

Abstract

The idea of dual sequence space was introduced by Köthe and Toeplitz [4], whose main results concerned $\alpha-$duals. An account of the duals of sequence spaces can be found in Köthe [3]. One can also known about different types of duals of sequence spaces in Maddox [7], Cook [16], Kamthan and Gupta [14] and many others. Recently Chandra and Tripathy [13] introduced the concept of generalized Köthe-Toeplitz duals of sequence spaces on introducing the concept of $\eta-$dual of sequence spaces. In this paper our main aim is to determine the $\eta-$dual of some difference sequence spaces. Further we establish some results involving the perfectness of difference sequence spaces relative to $\eta-$dual.

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1. Introduction

Let l_∞, c and c_0 be the linear spaces of bounded convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$||x||_\infty = \sup_k |x_k|$$

where $k \in N = \{1, 2, 3, ...\}$, the set of positive integers.

Kizmaz [5] defined the sequence spaces

$$l_\infty(\triangle) = \{x = (x_k) : \triangle\, x \in l_\infty\},$$

$$c(\triangle) = \{x = (x_k) : \triangle\, x \in c\},$$

$$c_0(\triangle) = \{x = (x_k) : \triangle\, x \in c_0\},$$

where $\triangle x = (\triangle x_k) = (x_k - x_{k+1})$, and showed that these are Banach spaces with norm

$$||x||_1 = |x_1| + ||\triangle x||_\infty.$$

The Colak [15] defined the sequence spaces $\triangle_v(X) = \{x = (x_k) : \triangle_v x_k \in X\}$, where $(\triangle_v x_k) = (v_k x_k - v_{k+1} x_{k+1})$ and $X = l_\infty, c$ or c_0, and investigated some topological properties of these spaces.

Further Et. and Colak [12] generalized the above sequence spaces to the following sequence spaces

$$l_\infty(\triangle^m) = \{x = (x_k) : \triangle^m\, x \in l_\infty\},$$

$$c(\triangle^m) = \{x = (x_k) : \triangle^m\, x \in c\},$$

$$c_0(\triangle^m) = \{x = (x_k) : \triangle^m\, x \in c_0\},$$

where $m \in N$, $\triangle^0 x = (x_k)$, $\quad \triangle x = (\triangle x_k) = (x_k - x_{k+1})$, $\triangle^m x = (\triangle^m x_k) = (\triangle^{m-1} x_k - \triangle^{m-1} x_{k+1})$, and

$$\triangle^m x_k = \sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} x_{k+j}$$

and showed that these are Banach spaces with norm

$$||x||_\triangle = \sum_{i=1}^{m} |x_i| + ||\triangle^m x||_\infty.$$

It is trivial that $c_0(\triangle^m) \subset c(\triangle^m) \subset l_\infty(\triangle^m)$ is satisfied and strict [12]. For convenience we denote these spaces $\triangle^m(l_\infty) = l_\infty(\triangle^m)$, $\triangle^m(c) = c(\triangle^m)$ and $\triangle^m(c_0) = c_0(\triangle^m)$.

After then Et. M. and Esi. A. [11] have defined the following sequence spaces :

$$\triangle_v^m(l_\infty) = \{x = (x_k) : \triangle_v^m\, x \in l_\infty\},$$

$$\triangle_v^m(c) = \{x = (x_k) : \triangle_v^m\, x \in c\},$$

$$\triangle_v^m(c_0) = \{x = (x_k) : \triangle_v^m x \in c_0\},$$

where $v = (v_k)$ be any fixed sequence of non-zero complex numbers, $m \in N$, $\triangle_v^0 x = (v_k x_k)$, $\triangle_v x_k = (\triangle v_k x_k) = (v_k x_k - v_{k+1} x_{k+1})$, $\triangle_v^m x = (\triangle_v^m x_k) = (\triangle_v^{m-1} x_k - \triangle_v^{m-1} x_{k+1})$, and so that

$$\triangle_v^m x_k = \sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} v_{k+j} x_{k+j}.$$

These are Banach spaces with norm

$$||x||_v = \sum_{i=1}^{m} |v_i x_i| + ||\triangle_v^m x||_\infty.$$

2. Main Results

In this section we find the η-dual of some difference sequence spaces. Also we show that these are not perfect spaces relative to $\eta-$dual.

Lemma (2.1). ([11]) $x \in \triangle_v^m(l_\infty)$ if and only if

(i) $\sup_k k^{-1} |\triangle_v^{m-1} x_k| < \infty$,

(ii) $\sup_k |\triangle_v^{m-1} x_k - k(k+1)^{-1} \triangle_v^{m-1} x_{k+1}| < \infty$.

Lemma (2.2). ([11]) $\sup_k k^{-1} |\triangle_v^{m-1} x_k| < \infty$ implies $\sup_k k^{-m} |v_k x_k| < \infty$.

Throughout the paper ϕ denote the space of all complex sequences.

Definition (2.3). ([13]) Let E be a non-empty subset of ϕ and $r \geq 1$, then the $\eta-$dual of E is define as

$$E^\eta = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k|^r < \infty \text{ for all } (x_k) \in E\}.$$

A non-empty subset E of ϕ is said to be perfect or $\eta-$reflexive if $E^{\eta\eta} = E$. Taking $r = 1$ in above definition we get $\alpha-$dual of E.

Lemma (2.4). ([13]) (i) E^η is linear subspace of ϕ for every $E \subset \phi$.

(ii) $E \subset F$ implies $E^\eta \supset F^\eta$ for $E, F \subset \phi$.

(iii) $E^{\eta\eta} = (E^\eta)^\eta \supset E$ for $E \subset \phi$.

(iv) $(\bigcup_j E_j)^\eta = \bigcap_j E_j^\eta$ for every family of $\{E_j\}$ with $E_j \subset \phi$ for all $j \in N$.

Theorem (2.5). Let $\mathcal{U}_1 = \{a = (a_k) : \sum_{k=1}^{\infty} k^{mr}|a_k v_k^{-1}|^r < \infty\}$, then

$$[\triangle_v^m(l_\infty)]^\eta = [\triangle_v^m(c)]^\eta = [\triangle_v^m(c_0)]^\eta = \mathcal{U}_1 \tag{1}$$

Proof. First we assume that $a \in \mathcal{U}_1$. Then

$$\sum_{k=1}^{\infty} k^{mr}|a_k v_k^{-1}|^r < \infty \tag{2}$$

Let $x \in \triangle_v^m(l_\infty)$. Then

$$\sum_{k=1}^{\infty} |a_k x_k|^r = \sum_{k=1}^{\infty} k^{mr}|a_k v_k^{-1}|^r (k^{-m}|v_k x_k|)^r$$

$$\leq \sum_{k=1}^{\infty} k^{mr}|a_k v_k^{-1}|^r < \infty.$$

for each $x \in \triangle_v^m(l_\infty)$, by Lemma 2.2. Thus we have to show that

$$\mathcal{U}_1 \subset [\triangle_v^m(l_\infty)]^\eta \tag{3}$$

Conversely let $a \notin \mathcal{U}_1$. Then for some k, we have

$$\sum_{k=1}^{\infty} k^{mr}|a_k v_k^{-1}| = \infty.$$

So we can find a sequence (n_i) of positive numbers n_i such that

$$\sum_{k=n_i+1}^{n(i+1)} k^{mr}|a_k v_k^{-1}|^r > i^r.$$

Now we define a sequence $x = (x_k)$ as

$$x_k = \begin{cases} 0, & 1 \leq k \leq n_1 \\ \dfrac{v_k^{-1}k^m}{i}, & (n_i + 1 < k \leq n(i+1) : i = 1, 2, 3, ...). \end{cases}$$

Then it is easy to verify that $x_k \in \triangle_v^m(c_0)$. But

$$\sum_{k=1}^{\infty} |a_k x_k|^r = \sum_{i=1}^{\infty} \Big\{ \sum_{k=n_i+1}^{n(i+1)} |a_k x_k|^r \Big\}$$

$$= \sum_{i=1}^{\infty} \Big\{ \sum_{k=n_i+1}^{n(i+1)} \frac{|a_k v_k^{-1}|^r \, k^{mr}}{i^r} \Big\}$$

$$> \sum_{i=1}^{\infty} 1 = \infty$$

$$\Rightarrow \qquad a_k \notin [\triangle_v^m(c_0)]^\eta,$$

and hence we have shown

$$[\triangle_v^m(c_0)]^\eta \subset \mathcal{U}_1. \tag{4}$$

Since $\triangle_v^m(c_0) \subset \triangle_v^m(c) \subset \triangle_v^m(l_\infty)$ implies $[\triangle_v^m(l_\infty)]^\eta \subset [\triangle_v^m(c)]^\eta \subset [\triangle_v^m(c_0)]^\eta$, (1) follows from (3) and (4).

Corollary (2.6). If we take $v_k = (1, 1, 1, ...)$ in theorem 2.5, then we obtain $\eta-$dual of sequence spaces defined by Et. and Colak [12] as

$$[\triangle^m(l_\infty)]^\eta = [\triangle^m(c)]^\eta = [\triangle^m(c_0)]^\eta = \{a = (a_k) : \sum_{k=1}^{\infty} k^{mr} |a_k|^r < \infty\}.$$

Corollary (2.7). If we take $v_k = (1, 1, ...)$ and $m = 2$ in theorem 2.5, then we obtain $\eta-$dual of sequence spaces defined by Et. [10] as

$$[\triangle^2(l_\infty)]^\eta = [\triangle^2(c)]^\eta = [\triangle^2(c_0)]^\eta = \{a = (a_k) : \sum_{k=1}^{\infty} k^{2r} |a_k|^r < \infty\}.$$

Corollary (2.8). If we take $v_k = (1, 1, 1, ...)$ and $m = 1$ in theorem 2.5, then we obtain $\eta-$dual of sequence spaces defined by Kizmaz [5] as

$$[\triangle(l_\infty)]^\eta = [\triangle(c)]^\eta = [\triangle(c_0)]^\eta = \{a = (a_k) : \sum_{k=1}^{\infty} k^{r} |a_k|^r < \infty\}.$$

Corollary (2.9). If we take $v_k = (1, 1, ...)$ and $m = 0$ in theorem 2.5, then we obtain $\eta-$dual of l_∞, c and c_0 as

$$(l_\infty)^\eta = (c)^\eta = (c_0)^\eta = l_r \qquad [2]$$

Theorem (2.10). Let $\mathcal{U}_2 = \{a = (a_k) : \sup_k k^{-mr} |a_k v_k|^r < \infty\}$, then

$$[\triangle_v^m(l_\infty)]^{\eta\eta} = [\triangle_v^m(c)]^{\eta\eta} = [\triangle_v^m(c_0)]^{\eta\eta} = \mathcal{U}_2 \tag{5}$$

Proof. First we assume that $a \in \mathcal{U}_2$. Then

$$\sup_k k^{-mr}|a_k v_k| < \infty. \tag{6}$$

Let $x \in [\triangle_v^m(c_0)]^\eta = \mathcal{U}_1$ by theorem 2.5. Then

$$\sum_{k=1}^{\infty} |a_k x_k|^r \leq \sup_k k^{-mr}|a_k v_k|^r \sum_{k=1}^{\infty} k^{mr}|x_k v_k^{-1}|^r < \infty.$$

By (6) and (1). Thus we have to show that

$$\mathcal{U}_2 \subset [\triangle_v^m(c_0)]^{\eta\eta} \tag{7}$$

Conversely suppose that $a \notin \mathcal{U}_2$. Then we have

$$\sup_k k^{-mr}|a_k v_k|^r = \infty.$$

Hence there is a strictly increasing sequence (k(i)) of positive integer $k(i)$, such that

$$[k(i)]^{-mr}|a_{k(i)}|^r > i^{mr}$$

We define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} |a_{k(i)}|^{-1}, & k = k(i) \\ 0, & k \neq k(i). \end{cases}$$

Then we have

$$\sum_{k=1}^{\infty} k^{mr}|x_k|^r = \sum_{i=1}^{\infty}[k(i)]^{mr}|a_{k(i)}|^{-r} \leq \sum_{i=1}^{\infty} i^{-mr} < \infty.$$

Hence $x \in [\triangle_v^m(l_\infty)]^\eta$ and $\sum_{k=1}^{\infty} |a_k x_k|^r = \infty$. Thus we have to show that

$$[\triangle_v^m(l_\infty)]^{\eta\eta} \subset \mathcal{U}_2 \tag{8}$$

Since $[\triangle_v^m(l_\infty)]^\eta \subset [\triangle_v^m(c)]^\eta \subset [\triangle_v^m(c_0)]^\eta$ implies $[\triangle_v^m(c_0)]^{\eta\eta} \subset [\triangle_v^m(c)]^{\eta\eta} \subset [\triangle_v^m(l_\infty)]^{\eta\eta}$, (5) follows from (7) and (8).

Corollary (2.11). If we take $v_k = (1, 1, 1, ...)$ in theorem 2.10, then we obtain second $\eta-$dual of the sequence spaces defined by Et. and Colak [12] as

$$[\triangle^m(l_\infty)]^{\eta\eta} = [\triangle^m(c)]^{\eta\eta} = [\triangle^m(c_0)]^{\eta\eta} = \{a = (a_k) : \sup_k k^{-mr}|a_k|^r < \infty\}.$$

If we take $m = 2, 1$ in corollary 2.11 then we obtain the second $\eta-$dual of sequence spaces defined in [10] and [5]. By theorem 2.5 and 2.10 we have the following corollaries :

Corollary (2.12). $\triangle_v^m(l_\infty)$, $\triangle_v^m(c)$ and $\triangle_v^m(c_0)$ are not perfect spaces relative to $\eta-$dual.

Corollary (2.13). $\triangle^m(l_\infty)$, $\triangle^m(c)$ and $\triangle^m(c_0)$ are not perfect spaces relative to $\eta-$dual.

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