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Regular Biclosure Spaces

Chawalit Boonpok

Department of Mathematics
Faculty of Science
Mahasarakham University
Mahasarakham 44150, Thailand
chawalit.b@msu.ac.th

Abstract

The purpose of this paper is to introduce the concept of regular biclosure spaces and investigate some of their characterizations.

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1 Introduction

J.C. Kelly [7] introduce the notion of bitopological spaces. Such spaces are equipped with two arbitrary topologies. Furthermore, Kelly extended some of the standard results of separation axioms in a topological space to a bitopological space. Thereafter, a large number of papers have been written to generalize topological concepts to bitopological setting. Closure spaces were introduced by E. Čech [3] and then studied by many authors, see e.g. [4, 5, 8, 9]. In this paper, we define and study the concept of regular biclosure spaces.

2 Preliminaries

A map $u : P(X) \to P(X)$ defined on the power set $P(X)$ of a set X is called a *closure operator* on X and the pair (X, u) is called a *closure space* if the following axioms are satisfied :

(N1) $u\emptyset = \emptyset$,

(N2) $A \subseteq uA$ for every $A \subseteq X$,

(N3) $A \subseteq B \Rightarrow uA \subseteq uB$ for all $A, B \subseteq X$.

A closure operator u on a set X is called *additive*(respectively, *idempotent*) if $A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB$ (respectively, $A \subseteq X \Rightarrow uuA = uA$). A subset $A \subseteq X$ is *closed* in the closure space (X, u) if $uA = A$ and it is *open* if its complement in X is closed. The empty set and the whole space are both open and closed. A closure space (Y, v) is said to be a *subspace* of (X, u) if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$. If Y is closed in (X, u), then the subspace (Y, v) of (X, u) is said to be closed too. Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$.

One can see that a map $f : (X, u) \to (Y, v)$ is continuous if and only if $uf^{-1}(B) \subseteq f^{-1}(vB)$ for every subset $B \subseteq Y$.

Clearly, if $f : (X, u) \to (Y, v)$ is continuous, then $f^{-1}(F)$ is a closed subset of (X, u) for every closed subset F of (Y, v).

Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *closed* (resp. *open*) if $f(F)$ is a closed (resp. open) subset of (Y, v) whenever F is a closed (resp. open) subset of (X, u).

The *product* of a family $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ of closure spaces, denoted by $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$, is the closure space $\big(\prod\limits_{\alpha \in I} X_\alpha, u\big)$ where $\prod\limits_{\alpha \in I} X_\alpha$ denotes the cartesian product of sets X_α, $\alpha \in I$, and u is the closure operator generated by the projections $\pi_\alpha : \prod\limits_{\alpha \in I} X_\alpha \to X_\alpha$, $\alpha \in I$, i.e., is defined by $uA = \prod\limits_{\alpha \in I} u_\alpha \pi_\alpha(A)$ for each $A \subseteq \prod\limits_{\alpha \in I} X_\alpha$.

Clearly, if $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ is a family of closure spaces, then the projection map $\pi_\beta : \prod\limits_{\alpha \in I}(X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$ is closed and continuous for every $\beta \in I$.

Proposition 2.1. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then F is a closed subset of (X_β, u_β) if and only if $F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$.*

Proof. Let F be a closed subset of (X_β, u_β). Since π_β is continuous, $\pi_\beta^{-1}(F)$ is a closed subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$. But $\pi_\beta^{-1}(F) = F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$, hence $F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$.

Conversely, let $F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ be a closed subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$. Since π_β is closed, $\pi_\beta\Big(F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha\Big) = F$ is a closed subset of (X_β, u_β). □

Proposition 2.2. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then G is an open subset of (X_β, u_β) if and only if $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is an open subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$.*

Definition 2.3. A *biclosure space* is a triple (X, u_1, u_2) where X is a set and u_1, u_2 are two closure operators on X.

Definition 2.4. A subset A of a biclosure space (X, u_1, u_2) is called *closed* if $u_1u_2A = A$. The complement of closed set is called *open*.

Clearly, A is a closed subset of a biclosure space (X, u_1, u_2) if and only if A is both a closed subset of (X, u_1) and (X, u_2).

Let A be a closed subset of a biclosure space (X, u_1, u_2) . The following conditions are equivalent

(i) $u_2u_1A = A$,

(ii) $u_1A = A$, $u_2A = A$.

Definition 2.5. Let (X, u_1, u_2) be a biclosure space. A biclosure space (Y, v_1, v_2) is called a *subspace* of (X, u_1, u_2) if $Y \subseteq X$ and $v_iA = u_iA \cap Y$ for each $i \in \{1, 2\}$ and each subset $A \subseteq Y$.

Proposition 2.6. *Let (X, u_1, u_2) be a biclosure space and let (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If F is a closed subset of (Y, v_1, v_2), then F is a closed subset of (X, u_1, u_2).*

Proof. Let F be a closed subset of (Y, v_1, v_2). Then $v_1F = F$ and $v_2F = F$. Since Y is both a closed subset of (X, u_1) and (X, u_2), $u_1F = F$ and $u_2F = F$. Consequently, F is both a closed subset of (X, u_1) and (X, u_2). Therefore, F is a closed subset of (X, u_1, u_2). □

Proposition 2.7. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then F is a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod_{\alpha \in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.*

Proof. Let $\beta \in I$ and let F be a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. Then F is a closed subset of (X_β, u^1_β) and (X_β, u^2_β), respectively. Since $\pi_\beta : \prod_{\alpha \in I}(X_\alpha, u^1_\alpha) \to (X_\beta, u^1_\beta)$ is continuous, $\pi^{-1}_\beta(F) = F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod_{\alpha \in I}(X_\alpha, u^1_\alpha)$. Similarly, since $\pi_\beta : \prod_{\alpha \in I}(X_\alpha, u^2_\alpha) \to (X_\beta, u^2_\beta)$ is continuous, $\pi^{-1}_\beta(F) = F \times$

$\prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha)$. Consequently, $F\times\prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.

Conversely, let $F\times\prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ be a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$. Then $F\times\prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$ and $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha)$, respectively. Since $\pi_\beta : \prod_{\alpha\in I}(X_\alpha, u^1_\alpha)\to(X_\beta, u^1_\beta)$ is closed, $\pi_\beta\Big(F\times\prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha\Big) = F$ is a closed subset of (X_β, u^1_β). Similarly, since $\pi_\beta : \prod_{\alpha\in I}(X_\alpha, u^2_\alpha)\to(X_\beta, u^2_\beta)$ is closed, $\pi_\beta\Big(F\times\prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha\Big) = F$ is a closed subset of (X_β, u^2_β). Consequently, F is a closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$. $\square$

Proposition 2.8. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha\in I\}$ be a family of biclosure spaces and let $\beta\in I$. Then G is an open subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $G\times\prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$.*

Definition 2.9. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $i\in\{1,2\}$. A map $f : (X, u_1, u_2)\to(Y, v_1, v_2)$ is called *i-closed* (resp. *i-open*) if the map $f : (X, u_i)\to(Y, v_i)$ is closed (resp. open). A map f is called *closed* (resp. *open*) if f is i-closed (resp. i-open) for each $i\in\{1,2\}$.

Definition 2.10. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $i\in\{1,2\}$. A map $f : (X, u_1, u_2)\to(Y, v_1, v_2)$ is called *i-continuous* if the map $f : (X, u_i)\to(Y, v_i)$ is continuous. A map f is called *continuous* if f is i-continuous for each $i\in\{1,2\}$.

3 Regular Biclosure Spaces

In this section, we introduce the concept of regular biclosuure spaces and study some of their properties.

Definition 3.1. A biclosure space (X, u_1, u_2) is said to be *regular biclosure space* if, for any closed subset F of (X, u_1) and any point $x\in X - F$, there exist open subsets U and V of (X, u_2) such that $x\in U$, $F\subseteq V$ and $U\cap V=\emptyset$.

Example 3.2. *Let $X=\{a,b\}$ and define a closure operator u_1 on X by $u_1\emptyset=\emptyset$, $u_1\{a\}=\{a\}$, $u_1\{b\}=\{b\}$ and $u_1X=X$. Define a closure operator u_2 on X by $u_2\emptyset=\emptyset$, $u_2\{a\}=\{a\}$, $u_2\{b\}=\{b\}$ and $u_2X=X$. Then (X, u_1, u_2) is regular biclosure space.*

Proposition 3.3. *Let (X, u_1, u_2) be a closure space. If (X, u_1, u_2) is regular biclosure space, then for any $x \in X$ and every open subset U of (X, u_1) containing x there exists an open subset V of (X, u_2) containing x such that $u_2 V \subseteq U$.*

Proof. Suppose that (X, u_1, u_2) is regular biclosure space and U is an open subset of (X, u_1) such that $x \in U$. Then $X - U$ is a closed subset of (X, u_1) which does not contain x. Therefore, there exist open subsets W and V of (X, u_2) such that $X - U \subseteq W$, $x \in V$ and $W \cap V = \emptyset$. Since $X - U \subseteq W$, we have $X - W \subseteq U$. Further $W \cap V = \emptyset$ implies $V \subseteq X - W \subseteq U$. But $X - W$ is a closed subset of (X, u_2) which contain V. Hence, $u_2 V \subseteq X - W \subseteq U$. $\square$

The following statement immediately follows Proposition 3.3:

Corollary 3.4. *Let (X, u_1, u_2) be a closure space. If (X, u_1, u_2) is regular biclosure space, then for any $x \in X$ and every closed subset F of (X, u_1) such that $x \notin F$, there exists an open subset G of (X, u_2) such that $x \in G$ and $u_2 G \cap F = \emptyset$.*

Lemma 3.5. *Let (X, u_1, u_2) be a biclosure space and let (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If G is an open subset of (X, u_1) and an open subset of (X, u_2), then $G \cap Y$ is an open subset of (Y, v_1) and an open subset of (Y, v_2).*

Proof. Let G be an open subset of (X, u_1). Since $Y - (G \cap Y) = Y \cap (X - G) = u_1 Y \cap u_1(X - G) = u_1(Y \cap (X - G)) = u_1(Y \cap (X - G)) \cap Y = v_1(Y \cap (X - G) = v_1(Y - (G \cap Y)$. Hence, $Y - (G \cap Y)$ is a closed subset of (Y, v_1). Consequently, $G \cap Y$ is an open subset of (Y, v_1). Similarly, if G is an open subset of (X, u_2), then $G \cap Y$ is an open subset of (Y, v_2) $\square$

Lemma 3.6. *Let (X, u_1, u_2) be a biclosure space and let (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If F is a closed subset of (Y, v_1, v_2), then F is a closed subset of (X, u_1, u_2).*

Proof. Let F be a closed subset of (Y, v_1, v_2). Then F is both a closed subset of (Y, v_1) and (Y, v_2). Consequently, $F = v_1 F = u_1 F \cap Y = u_1 F \cap u_1 Y = u_1(F \cap Y) = u_1 F$ and $F = u_2 F$. Therefore, F is both a closed subset of (X, u_1) and (X, u_2). Hence, F is a closed subset of (X, u_1, u_2). $\square$

Proposition 3.7. *Let (X, u_1, u_2) be a biclosure space and let (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If (X, u_1, u_2) is regular biclosure space, then (Y, v_1, v_2) is regular biclosure space.*

Proof. Let F be a closed subset of (Y, v_1) such that $y \notin F$. By Lemma 3.6, F is a closed subset of (X, u_1) such that $y \notin F$. Since (X, u_1, u_2) is regular biclosure space, there exist open subsets U and V of (X, u_2) such that $y \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. Consequently, $y \in U \cap Y$ and $F \subseteq V \cap Y$. By Lemma 3.5, $U \cap Y$ and $V \cap Y$ are open subsets of (Y, v_2) such that $(U \cap Y) \cap (V \cap Y) = \emptyset$. Hence, (Y, v_1, v_2) is regular biclosure space. $\square$

Proposition 3.8. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces. Then $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is regular biclosure space if and only if $(X_\alpha, u^1_\alpha, u^2_\alpha)$ is regular biclosure space for each $\alpha \in I$.*

Proof. Suppose that $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is regular biclosure space. Let $\beta \in I$ and let F be a closed subset of (X, u^1_β) such that $x_\beta \notin F$. Then $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$ such that $(x_\alpha)_{\alpha\in I} \notin F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Since $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is regular biclosure space, there exist open subsets U and V of (X_β, u^2_β) such that $x_\beta \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. Hence, $(X_\beta, u^1_\beta, u^2_\beta)$ is regular biclosure space.

Conversely, suppose that $(X_\alpha, u^1_\alpha, u^2_\alpha)$ is regular biclosure space for each $\alpha \in I$. Let F be a closed subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha)$ such that $(x_\alpha)_{\alpha\in I} \notin F$. Then $\pi_\beta(F)$ is a closed subset of (X_β, u^1_β) such that $x_\beta \notin \pi_\beta(F)$. Since $(X_\beta, u^1_\beta, u^2_\beta)$ is regular biclosure space, there exists open subsets U and V of (X_β, u^2_β) such that $x_\beta \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. Therefore, $(x_\alpha)_{\alpha\in I} \in U \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ and $F \subseteq \pi^{-1}_\beta(V) = V \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$. Consequently, $U \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ and $V \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ are open subsets of $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha)$ such that $(U \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha) \cap (V \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha) = \emptyset$. Hence, $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is regular biclosure space. □

Proposition 3.9. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be injective, closed and continuous. If (Y, v_1, v_2) is regular biclosure space, then (X, u_1, u_2) is regular biclosure space.*

Proof. Let F be a closed subset of (X, u_1) such that $x \notin F$. Then $f(F)$ is a closed subset of (Y, v_1) such that $f(x) \notin f(F)$. Since (Y, v_1, v_2) is regular biclosure space, there exist open subset U and V of (Y, v_2) such that $f(x) \in U$, $f(F) \subseteq V$ and $U \cap V = \emptyset$. Consequently, $f^{-1}(U)$ and $f^{-1}(V)$ are open subsets of (X, u_2) such that $x \in f^{-1}(U)$, $F \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence, (X, u_1, u_2) is regular biclosure space. □

References

[1] C. Boonpok, *C_0-Spaces and C_1-Spaces*, Acta Univ. Apulensis, **17** (2009), 37–42.

[2] C. Boonpok, *$T_{\frac{1}{2}}$-Biclosure Spaces*, Int. Math. Forum, **5** (2)(2010), 51–58.

[3] E. Čech, *Topological Spaces*, Topological Papers of Eduard Čech, Academia, Prague 1968, 436–472.

[4] J. Chvalina, *On homeomorphic topologies and equivalent set-systems*, Arch. Math. 2, Scripta Fac. Sci. Nat. UJEP Brunensis, **XII** (1976), 107–116.

[5] J. Chvalina, *Stackbases in power sets of neighbourhood spaces preserving the continuity of mappings*, Arch. Math. 2, Scripta Fac. Sci. Nat. UJEP Brunensis, **XVII** (1981), 81–86.

[6] J. Dujunji, *Topology*, Allyn and Bacon, Boston, 1966.

[7] J.C. Kelly, *Bitopological spaces*, Pro. London Math. Soc. **3** (13)(1969), 71–79.

[8] L. Skula, *Systeme von stetigen abbildungen*, Czech. Math. J., **17** (92) (1967), 45–52.

[9] J. Šlapal, *Closure operations for digital topology*, Theoret. Comput. Sci., **305** (2003), 457–471.

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