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Hausdorff Biclosure Spaces

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Abstract

The purpose of this paper is to introduce the concept of Hausdorff biclosure spaces and investigate some of their characterizations.

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1 Introduction

Closure spaces were introduced by E. Čech [3] and then studied by many authors, see e.g. [4, 5, 7, 8]. J.C. Kelly [6] introduce the notion of bitopological spaces. Such spaces are equipped with two arbitrary topologies. Furthermore, Kelly extended some of the standard results of separation axioms in a topological space to a bitopological space. Thereafter, a large number of papers have been written to generalize topological concepts to bitopological setting. In this paper, we introduce the concept of Hausdorff biclosure spaces and study some of their properties.

2 Preliminaries

A map $u : P(X) \to P(X)$ defined on the power set $P(X)$ of a set X is called a *closure operator* on X and the pair (X, u) is called a *closure space* if the following axioms are satisfied :

(N1) $u\emptyset = \emptyset$,

(N2) $A \subseteq uA$ for every $A \subseteq X$,

(N3) $A \subseteq B \Rightarrow uA \subseteq uB$ for all $A, B \subseteq X$.

A closure operator u on a set X is called *additive*(respectively, *idempotent*) if $A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB$ (respectively, $A \subseteq X \Rightarrow uuA = uA$). A subset $A \subseteq X$ is *closed* in the closure space (X, u) if $uA = A$ and it is *open* if its complement in X is closed. The empty set and the whole space are both open and closed. A closure space (Y, v) is said to be a *subspace* of (X, u) if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$. If Y is closed in (X, u), then the subspace (Y, v) of (X, u) is said to be closed too. Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$.

One can see that a map $f : (X, u) \to (Y, v)$ is continuous if and only if $uf^{-1}(B) \subseteq f^{-1}(vB)$ for every subset $B \subseteq Y$.

Clearly, if $f : (X, u) \to (Y, v)$ is continuous, then $f^{-1}(F)$ is a closed subset of (X, u) for every closed subset F of (Y, v).

Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *closed* (resp. *open*) if $f(F)$ is a closed (resp. open) subset of (Y, v) whenever F is a closed (resp. open) subset of (X, u).

The *product* of a family $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ of closure spaces, denoted by $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$, is the closure space $\big(\prod\limits_{\alpha \in I} X_\alpha, u\big)$ where $\prod\limits_{\alpha \in I} X_\alpha$ denotes the cartesian product of sets X_α, $\alpha \in I$, and u is the closure operator generated by the projections $\pi_\alpha : \prod\limits_{\alpha \in I} X_\alpha \to X_\alpha$, $\alpha \in I$, i.e., is defined by $uA = \prod\limits_{\alpha \in I} u_\alpha \pi_\alpha(A)$ for each $A \subseteq \prod\limits_{\alpha \in I} X_\alpha$.

Clearly, if $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ is a family of closure spaces, then the projection map $\pi_\beta : \prod\limits_{\alpha \in I}(X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$ is closed and continuous for every $\beta \in I$.

Proposition 2.1. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then F is a closed subset of (X_β, u_β) if and only if $F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$.*

Proof. Let F be a closed subset of (X_β, u_β). Since π_β is continuous, $\pi_\beta^{-1}(F)$ is a closed subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$. But $\pi_\beta^{-1}(F) = F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$, hence $F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$.

Conversely, let $F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ be a closed subset of $\prod\limits_{\alpha \in I}(X_\alpha, u_\alpha)$. Since π_β is closed, $\pi_\beta\Big(F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha\Big) = F$ is a closed subset of (X_β, u_β). □

Proposition 2.2. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then G is an open subset of (X_β, u_β) if and only if $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is an open subset of $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$.*

Definition 2.3. A *biclosure space* is a triple (X, u_1, u_2) where X is a set and u_1, u_2 are two closure operators on X.

Definition 2.4. A subset A of a biclosure space (X, u_1, u_2) is called *closed* if $u_1u_2A = A$. The complement of closed set is called *open*.

Clearly, A is a closed subset of a biclosure space (X, u_1, u_2) if and only if A is both a closed subset of (X, u_1) and (X, u_2).

Let A be a closed subset of a biclosure space (X, u_1, u_2) . The following conditions are equivalent

(i) $u_2u_1A = A$,

(ii) $u_1A = A$, $u_2A = A$.

Definition 2.5. Let (X, u_1, u_2) be a biclosure space. A biclosure space (Y, v_1, v_2) is called a *subspace* of (X, u_1, u_2) if $Y \subseteq X$ and $v_iA = u_iA \cap Y$ for each $i \in \{1, 2\}$ and each subset $A \subseteq Y$.

Proposition 2.6. *Let (X, u_1, u_2) be a biclosure space and let (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If F is a closed subset of (Y, v_1, v_2), then F is a closed subset of (X, u_1, u_2).*

Proof. Let F be a closed subset of (Y, v_1, v_2). Then $v_1F = F$ and $v_2F = F$. Since Y is both a closed subset of (X, u_1) and (X, u_2), $u_1F = F$ and $u_2F = F$. Consequently, F is both a closed subset of (X, u_1) and (X, u_2). Therefore, F is a closed subset of (X, u_1, u_2). □

Definition 2.7. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $i \in \{1, 2\}$. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called *i-continuous* if the map $f : (X, u_i) \to (Y, v_i)$ is continuous. A map f is called *continuous* if f is i-continuous for each $i \in \{1, 2\}$.

3 Hausdorff Biclosure Spaces

In this section, we introduce the concept of Hausdorff biclosuure spaces and study some of their properties.

Definition 3.1. A biclosure space (X, u_1, u_2) is called a *Hausdorff biclosure space* if, whenever x and y are distinct points of X there exists an open subset U of (X, u_1) and an open subset V of (X, u_2) such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Example 3.2. *Let $X = \{a, b\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{a\} = \{a\}$, $u_1\{b\} = \{b\}$ and $u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{a\} = \{a\}$, $u_2\{b\} = \{b\}$ and $u_2X = X$. Then (X, u_1, u_2) is a Hausdorff biclosure space.*

Lemma 3.3. *Let (X, u_1, u_2) be a biclosure space and let (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If G is an open subset of (X, u_1) and an open subset of (X, u_2), then $G \cap Y$ is an open subset of (Y, v_1) and an open subset of (Y, v_2).*

Proof. Let G be an open subset of (X, u_1). Since $Y-(G\cap Y) = Y\cap(X-G) = u_1Y\cap u_1(X-G) = u_1(Y\cap(X-G)) = u_1(Y\cap(X-G))\cap Y = v_1(Y\cap(X-G)) = v_1(Y-(G\cap Y))$. Hence, $Y-(G\cap Y)$ is a closed subset of (Y, v_1). Consequently, $G\cap Y$ is an open subset of (Y, v_1). Similarly, if G is an open subset of (X, u_2), then $G \cap Y$ is an open subset of (Y, v_2) □

Proposition 3.4. *Let (X, u_1, u_2) be a biclosure space and let (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If (X, u_1, u_2) is a Hausdorff biclosure space, then (Y, v_1, v_2) is a Hausdorff biclosure space.*

Proof. Let y and y' be any two distinct points of Y. Then y and y' are distinct points of X. Since (X, u_1, u_2) is a Haurdorff biclosure space, there exists a disjoint open subset U of (X, u_1) and an open subset V of (X, u_2) containing y and y', respectively. Consequently, $y \in U \cap Y$, $y' \in V \cap Y$ and $(U \cap Y) \cap (V \cap Y) = \emptyset$. By Lemma 3.3, $U \cap Y$ is an open subset of (Y, v_1) and $V \cap Y$ is an open subset of (Y, v_2). Hence, (Y, v_1, v_2). is a Hausdorff biclosure space. □

Proposition 3.5. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces. Then $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a Hausdorff biclosure space if and only if $(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a Hausdorff biclosure space for each $\alpha \in I$.*

Proof. Suppose that $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a Hausdorff biclosure space. Let $\beta \in I$ and x_β, y_β be any two distinct points of X_β. Then $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ are distinct points of $\prod_{\alpha\in I} X_\alpha$. Since $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a Hausdorff biclosure space. There exists an open subset U of (X_β, u^1_β) and an open subset V of (X_β, u^2_β) such that $x_\beta \in U$, $y_\beta \in V$ and $U\cap V = \emptyset$. Therefore, $(X_\beta, u^1_\beta, u^2_\beta)$ is a Hausdorff biclosure space.

Conversely, suppose that $(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a Hausdorff biclosure space for each $\alpha \in I$. Let $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ be any two distinct points of $\prod_{\alpha\in I} X_\alpha$. Then x_β and y_β are distinct points of X_β. Since $(X_\beta, u^1_\beta, u^2_\beta)$ is a Hausdorff biclosure space, there exists a disjoint open subset U of (X_β, u^1_β) and an open subset V of (X_β, u^2_β) such that $x_\beta \in U$ and $y_\beta \in V$. Consequently, $U \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha,)$ and $V\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u^2_\alpha)$ such that $(x_\alpha)_{\alpha\in I}\in U\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$, $(y_\alpha)_{\alpha\in I}\in V\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ and $(U\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha)\cap(V\times\prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha)=\emptyset$. Hence, $\prod_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a Hausdorff biclosure space. □

Proposition 3.6. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. Let $f : (X, u_1, u_2)\to(Y, v_1, v_2)$ be injective and continuous. If (Y, v_1, v_2) is a Hausdorff biclosure space, then (X, u_1, u_2) is a Hausdorff biclosure space.*

Proof. Let x and y be any two distinct points of X. Then $f(x)$ and $f(y)$ are distinct points of Y. Since (Y, v_1, v_2) is a Hausdorff biclosure space, there exists a disjoint open subset U of (Y, v_1) and an open subset V of (Y, v_2) containing $f(x)$ and $f(y)$, respectively. Since f is continuous and $U\cap V=\emptyset$, $f^{-1}(U)$ is an open subset of (X, u_1), $f^{-1}(V)$ is an open subset of (X, u_2), $f^{-1}(U)\cap f^{-1}(V)=\emptyset$ and $x\in f^{-1}(U)$, $y\in f^{-1}(V)$. Therefore, (X, u_1, u_2) is a Hausdorff biclosure space. □

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