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Homothetic Motions at $\mathbf{E}^4_{\alpha\beta}$

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Abstract

In this paper, a matrix corresponding to Hamilton operators is defined for generalized quaternions is detemined a Hamilton motion in four-dimensional space $E^4_{\alpha\beta}$. It is shown that this is a homothetic motion. Also, it is found that the Hamilton motion defined by a regular curve of order r has only one acceleration center of order $(r-1)$ at every instant t.

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1 Introduction

To investigate the geometry of the motion of a line or a point in the motion of space is important in the study of space kinematices or spatial mechanisms or in physics. The geometry of such a motion of a point or a line has a number of applications in geometric modeling and model-based manufacturing of mechanical products or in the design of robotic motions. Hacısalihoğlu[3] showed some properties of 1-parameter homothetic motion in Euclidean space E^n. In addition, he found that this motion is regular and has one pole point at every t-instant. After him, Yaylı[7] gave homothetic motions with aid of the Hamilton operators in four-dimensional Euclidean space E^4. Subsequently, Kula and Yaylı[5] expressed Hamilton motions by means of Hamilton operators in semi-Euclidean space E^4_2 and showed that this motions, are a homothetic motion. Also, this subject is investigated in algebra[2]. Recently, we studied the generalized quaternions, and presented some of their algebraic properties[4]. Furthermore, we give some algebraic properties of Hamilton operators of generalized quaternion. In [4], generalized quaternions have expressed in terms of 4×4 matrices by means of these operators. In this paper, first, we define a motion by using these matrices, and show that this motion is a homothetic motion in four-dimensional space $E^4_{\alpha\beta}$. We find that the homothetic motion has only one pole point at every instant t, and prove that this motion has only one acceleration center of high order at every instant t.

2 Preliminaries

Definition 1. *A generalized quaternion q is defined as*

$$q = a_. + a_1 i + a_2 j + a_3 k$$

where $a_., a_1, a_2$ and a_3 are real numbers and $1, i, j, k$ of q may be interpreted as the four basic vectors of cartesian set of coordinates; and they satisfy the non-commutative multiplication rules

$$\begin{aligned} i^2 &= -\alpha, \quad j^2 = -\beta, \quad k^2 = -\alpha\beta \\ ij &= k = -ji, \quad jk = \beta i = -kj \end{aligned}$$

and

$$ki = \alpha j = -ik, \qquad \alpha, \beta \in \mathbb{R}.$$

The set of all generalized quaternions are denoted by $H_{\alpha\beta}$. So, a generalized quaternion q is a sum of a scalar and a vector, called scalar part, $S_q = a_.$, and vector part $V_q = a_1 i + +a_2 j + a_3 k \in \mathbb{R}^3_{\alpha\beta}$. Therefore, $H_{\alpha\beta}$ is form a 4-dimensional real space which contains the real axis $\mathbb{R}$ and a 3-dimensional real linear space $\mathbb{R}^3_{\alpha\beta}$, so that, $H_{\alpha\beta} = \mathbb{R} \oplus \mathbb{R}^3_{\alpha\beta}$. It is clear, if $\alpha = \beta = 1$ then $H_{\alpha\beta} = H$ (real quaternions), and if $\alpha = 1$, $\beta = -1$ then $H_{\alpha\beta} = H'$(split quaternions) [4].

Definition 2. *We define a generalized inner product in* $\mathbb{R}^4$,

$$\langle u, v \rangle = u_1 v_1 + \alpha u_2 v_2 + \beta u_3 v_3 + \alpha\beta u_4 v_4$$

where $u = (u_1, u_2, u_3, u_4)$, $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$ and $\alpha, \beta \in \mathbb{R}$. We put $E^4_{\alpha\beta} = (\mathbb{R}^4, \langle , \rangle)$. So, we identity $H_{\alpha\beta}$ with the 4-dimensional space $E^4_{\alpha\beta}$.

Definition 3. *A matrix A is called a quasi-orthogonal matrix if $A^T \varepsilon A = \varepsilon$ and $det A = 1$, where* $\varepsilon = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha\beta \end{bmatrix}$ *and* $\alpha, \beta \in \mathbb{R}$ [4].

3 Homothetic motions in $E^4_{\alpha\beta}$

The 1-parameter homothetic motions of a body in four-dimensional space $E^4_{\alpha\beta}$ is generated by transformation

$$\begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} hA & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix}$$

where A is a quasi-orthogonal matrix. The matrix $B = hA$ is called a homothetic matrix and Y, X and C are $n \times 1$ real matrices. The homothetic scalar h and the elements of A and C are continuously differentiable functions of a real parameter t. Y and X correspond to the position vectors of the same point with respect to the rectangular coordinate systems of the moving space R and the fixed space $R_\circ$,respectively. At the initial time $t = t_\circ$, we consider the coordinate systems of R and $R_\circ$ as coincident. To avoid the case of affine transformation we assume that

$$h = h(t) \neq cons.\ ,\ h(t) \neq 0.$$

and to avoid the case of a pure translation or a pure rotation, we also assume that

$$\frac{d}{dt}(hA) \neq 0, \quad \frac{d}{dt}(C) \neq 0.$$

4 Hamilton motions in $\mathbf{E}^4_{\alpha\beta}$

Let $q = a_. + a_1 i + a_2 j + a_3 k$ be a generalized quaternion, and let $h_q : H_{\alpha\beta} \to H_{\alpha\beta}$, $h_q(x) = qx$. The matrix of h_q relative to the natural basis $\{1, i, j, k\}$ for $H_{\alpha\beta}$ is

$$H(q) = \begin{bmatrix} a_. & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_. & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_. & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_. \end{bmatrix} \quad (1)$$

(*see* [4]).

Let us consider the following curve:

$$\begin{aligned} \mathbf{a} &: I \subset \mathbb{R} \to E^4_{\alpha\beta} \\ \mathbf{a}(t) &= [a_.(t), a_1(t), a_2(t), a_3(t)],\ \forall t \in I \end{aligned}$$

we suppose that the unit velocity curve $\mathbf{a}(t)$ is differentiable regular curve of order r. The operator B called the generalized Hamiltonian operator, corresponding to $\mathbf{a}(t)$ is defined by the following matrix;

$$B = H[\mathbf{a}(t)] = \begin{bmatrix} a_.(t) & -\alpha a_1(t) & -\beta a_2(t) & -\alpha\beta a_3(t) \\ a_1(t) & a_.(t) & -\beta a_3(t) & \beta a_2(t) \\ a_2(t) & \alpha a_3(t) & a_.(t) & -\alpha a_1(t) \\ a_3(t) & -a_2(t) & a_1(t) & a_.(t) \end{bmatrix}. \qquad (2)$$

Definition 4. *The 1-parameter Hamilton motions of a body in $E^4_{\alpha\beta}$ are generated by transformation*

$$\begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix}$$

or equivalently

$$Y = BX + C. \qquad (3)$$

Here $B = H[\mathbf{a}(t)]$ and Y, X and C are $n \times 1$ real matrices, A and C are continuously differentiable functions of a real parameter t; Y and X correspond to the position vectors of the same point P.

Theorem 1. *The Hamilton motion determined by equation (3) is a homothetic motion in $E^4_{\alpha\beta}$.*

Proof. We suppose that length of $\mathbf{a}(t)$ is not zero, so the matrix B can be represented as

$$B = h \begin{bmatrix} \frac{a_.(t)}{h} & \frac{-\alpha a_1(t)}{h} & \frac{-\beta a_2(t)}{h} & \frac{-\alpha\beta a_3(t)}{h} \\ \frac{a_1(t)}{h} & \frac{a_.(t)}{h} & \frac{-\beta a_3(t)}{h} & \frac{\beta a_2(t)}{h} \\ \frac{a_2(t)}{h} & \frac{\alpha a_3(t)}{h} & \frac{a_.(t)}{h} & \frac{-\alpha a_1(t)}{h} \\ \frac{a_3(t)}{h} & \frac{-a_2(t)}{h} & \frac{a_1(t)}{h} & \frac{a_.(t)}{h} \end{bmatrix} = hA \qquad (4)$$

where $h : I \subset \mathbb{R} \to \mathbb{R}$,

$$t \to h(t) = \|\mathbf{a}(t)\| = \sqrt{|a_.^2(t) + \alpha a_1^2(t) + \beta a_2^2(t) + \alpha\beta a_3^2(t)|}$$

so, we find $A^T \varepsilon A = \varepsilon$ and $\det A = 1$, thus B is a homothetic matrix and equation (3) determines a homothetic motion. □

Special cases:

(i) For the case $\alpha = \beta = 1$, A is a orthogonal matrix and equation (3) determines a homothetic motion at E^4.(*see*[7])

(ii) For the case $\alpha = 1, \beta = -1$, A is a semi-orthogonal matrix and equation (3) determines a homothetic motion in semi-Euclidean space E^4_2. (*see*[5]).

Theorem 2. *The derivation operator $\dot{B}$ of the Hamilton operator $B = hA$, is a quasi-orthogonal matrix.*

Proof. We derivate of (2), i.e. $\dot{B} = H[\dot{\mathbf{a}}(t)]$, we have $\dot{B}^T \varepsilon \dot{B} = \varepsilon$, and since $\mathbf{a}(t)$ is unit velocity curve then $\det \dot{B} = 1$. □

Theorem 3. *In $E^4_{\alpha\beta}$, the Hamilton motion is a regular motion, and it does not depend on h.*

If we differentiate of (3) with respect to t yields

$$\dot{Y} = \dot{B}X + \dot{C} + B\dot{X},$$

where

$$V_r = B\dot{X}$$

is the relative velocity of X, $V_s = \dot{B}X + \dot{C}$ is the sliding velocity of X and $V_a = \dot{Y}$ is called absolute velocity of point X. So, we can give the following theorem.

Theorem 4. *In four-dimensional space $E^4_{\alpha\beta}$, for 1-parameter homothetic motion, absolute velocity vector of moving system of a point X at time t is the sum of the sliding velocity vector and relative velocity vector of that point.*

5 Pole points and pole curves of the motion

We look for points where the sliding velocity of the motion is zero at all time t, such points are called pole points of the motion at that instant in $R_\circ$. Hence,

$$\dot{B}X + \dot{C} = 0. \quad (5)$$

by theorem 4.2, $\dot{B}$ is regular, so equation (5) has only one solution, i.e.

$$X = -\dot{B}^{-1}\dot{C}$$

at every instant t. In this case the following theorem can be given.

Theorem 5. *The pole point corresponding to each instant t in $R_\circ$ is the rotation by $\dot{B}^{-1}$ of the speed vector $\dot{C}$ of the translation vector at that moment.*

Proof. As the matrix $\dot{B}$ is quasi-orthogonal, the matrix $\dot{B}^{-1}$ is quasi-orthogonal too. Thus, it makes a rotation. □

Theorem 6. *During the homothetic motion the pole curves slide and roll upon each others and the number of the sliding-rolling of the motion is h.*

6 Acceleration centers of order $(r-1)$ of the motion

Definition 5. *The set of zeros of the equation of the sliding acceleration of order r is called the acceleration center of order (r-1)[7].*

In order to find the acceleration center of order (r-1) for the equation (3) according to definition above, we find the solution of the equation

$$B^{(r)}X + C^{(r)} = 0, \qquad (6)$$

where

$$B^{(r)} = \frac{d^r B}{dt^r}, \quad C^{(r)} = \frac{d^r C}{dt^r}.$$

As the curve $\mathbf{a}(t)$ is a regular curve of order r, then

$$\left(a_\circ^{(r)}(t)\right)^2 + \alpha\left(a_1^{(r)}(t)\right)^2 + \beta\left(a_2^{(r)}(t)\right)^2 + \alpha\beta\left(a_3^{(r)}(t)\right)^2 \neq 0, \quad a_i^{(r)} = \frac{d^r a_i}{dt^i},$$

Also, as

$$\det B^{(r)} = \left\{ \left[a_\cdot^{(r)}\right]^2 + \alpha\left[a_1^{(r)}\right]^2 + \beta\left[a_2^{(r)}\right]^2 + \alpha\beta\left[a_3^{(r)}\right]^2 \right\}^2,$$

then $\det B^{(r)} \neq 0$. Therefore matrix $B^{(r)}$ has an inverse, and, by equation (6), the acceleration center of order $(r-1)$ at every t instant, is

$$X = \left[B^{(r)}\right]^{-1}\left(-C^{(r)}\right).$$

Example 1. *Let* $\mathbf{a}: I \subset \mathbb{R} \to E^4_{\alpha\beta}$ *be a curve given by*

$$t \to \mathbf{a}(t) = \frac{1}{\sqrt{2}}\left(\cos t, \frac{1}{\sqrt{\alpha}}\sin t, \frac{1}{\sqrt{\beta}}\cos t, \frac{1}{\sqrt{\alpha\beta}}\sin t\right), \ \alpha, \beta \gneqq 0.$$

$\mathbf{a}(t)$ is a unit velocity curve and differentiable regular of order r. Matrix B can be represented as

$$B = H[\mathbf{a}(t)] = \frac{1}{\sqrt{2}}\begin{bmatrix} \cos t & -\sqrt{\alpha}\sin t & -\sqrt{\beta}\cos t & -\sqrt{\alpha\beta}\sin t \\ \frac{1}{\sqrt{\alpha}}\sin t & \cos t & -\sqrt{\frac{\alpha}{\beta}}\sin t & \sqrt{\beta}\cos t \\ \frac{1}{\sqrt{\beta}}\cos t & \sqrt{\frac{\beta}{\alpha}}\sin t & \cos t & -\sqrt{\alpha}\sin t \\ \frac{1}{\sqrt{\alpha\beta}}\sin t & -\frac{1}{\sqrt{\beta}}\cos t & \frac{1}{\sqrt{\alpha}}\sin t & \cos t \end{bmatrix}$$

Thus $\mathbf{a}(t)$ satisfies all conditions of the above theorems.

let $C = (0, t, 0, 0)$, the (3) motion is given by

$$Y = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos t & -\sqrt{\alpha}\sin t & -\sqrt{\beta}\cos t & -\sqrt{\alpha\beta}\sin t \\ \frac{1}{\sqrt{\alpha}}\sin t & \cos t & -\sqrt{\frac{\alpha}{\beta}}\sin t & \sqrt{\beta}\cos t \\ \frac{1}{\sqrt{\beta}}\cos t & \sqrt{\frac{\beta}{\alpha}}\sin t & \cos t & -\sqrt{\alpha}\sin t \\ \frac{1}{\sqrt{\alpha\beta}}\sin t & -\frac{1}{\sqrt{\beta}}\cos t & \frac{1}{\sqrt{\alpha}}\sin t & \cos t \end{bmatrix} X + \begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix}. \quad (7)$$

Hence geometrical path of pole points in the Hamilton motion is determined by equation (7) as

$$X = \dot{B}^{-1}(-\dot{C}) = \varepsilon^{-1}\dot{B}^{T}\varepsilon(-\dot{C})$$

$$X = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\beta}\sin t \\ -(\frac{\beta}{\alpha})^{\frac{3}{2}}\cos t \\ \sin t \\ \frac{1}{\sqrt{\alpha}}\cos t \end{bmatrix}.$$

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