A Note on Global Behaviour of Solutions and Positive Nonoscillatory Solution

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Abstract

In this paper we study the global behaviours of solutions of the fourth-order difference equation

$$x_{n+1} = \frac{x_n x_{n-1} x_{n-2} x_{n-3} + f(x_n, x_{n-1}, x_{n-2}, x_{n-3}) x_{n-3}^{k+1} + x_{n-3}^m + a}{x_n x_{n-1} x_{n-2} + f(x_n, x_{n-1}, x_{n-2}, x_{n-3}) x_{n-3}^k + x_{n-3}^m + a},$$

$n = 0, 1, 2, ...$, where $f : (0, \infty)^4 \to (0, \infty)$ is arbitrary and differentiable, $m, k \in [0, \infty)$ and the initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$. We also study the positive nonoscillatory solutions of the following difference equation

$$x_{n+1} = \frac{A_1}{x_n + x_{n-1}} + \frac{A_2}{x_{n-1} + x_{n-2}} + ... + \frac{A_{k-1}}{x_{n-k+2} + x_{n-k+1}} - \frac{1}{x_{n-k}},$$

$n = 0, 1, 2, ...$, where $A_1, A_2, ..., A_{k-1} \in [0, \infty)$ and $A = \sum_{i=1}^{k-1} A_i - 2 > 0$.

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1   Introduction

It is extremely difficult to understand throughly the global behaviours of so-
lutions of rational difference equations although they have simple forms. We
can refer to [1-3, 7-12], especially [1, 3] for examples to illustrate this.

   The study of rational difference equations of order greater than one is quite
challenging and rewarding because some prototypes for the development of the
basic theory of the global behaviour of nonlinear difference equations of order
greater than one come from the results for rational difference equations.

   G. Ladas proposed to study the rational difference equation

$$x_{n+1} = \frac{x_n + x_{n-1}x_{n-2} + a}{x_n x_{n-1} + x_{n-2} + a}, \quad n = 0, 1, 2, ... \tag{1}$$

From then on, rational difference equations with the unique positive equilib-
rium $\bar{x} = 1$ have received considerable attention, one can refer to [7-11] and
the references cited therein.

   Recently, Li [9] investigated the global behaviour of the following fourth-
order rational difference equation

$$x_{n+1} = \frac{x_n x_{n-1} x_{n-3} + x_n + x_{n-1} + x_{n-3} + a}{x_n x_{n-1} + x_n x_{n-3} + x_{n-1} x_{n-3} + 1 + a}, \quad n = 0, 1, 2, ... \tag{2}$$

where $a \in [0, \infty)$ and initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$.

   In this note, we consider the following fourth-order rational difference equa-
tion

$$x_{n+1} = \frac{x_n x_{n-1} x_{n-2} x_{n-3} + f(x_n, x_{n-1}, x_{n-2}, x_{n-3})x_{n-3}^{k+1} + x_{n-3}^m + a}{x_n x_{n-1} x_{n-2} + f(x_n, x_{n-1}, x_{n-2}, x_{n-3})x_{n-3}^k + x_{n-3}^m + a}, \tag{3}$$

$n = 0, 1, 2, ...$, where $f : (0, \infty)^4 \rightarrow (0, \infty)$ is arbitrary and differentiable,
$m, k, a \in [0, \infty)$ and the initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$.
It is easy to see that the positive equilibrium $\bar{x}$ of Eq. (3) satisfies

$$\bar{x} = \frac{\bar{x}^4 + f(\bar{x}, \bar{x}, \bar{x}, \bar{x})\bar{x}^{k+1} + \bar{x}^m + a}{\bar{x}^3 + f(\bar{x}, \bar{x}, \bar{x}, \bar{x})\bar{x}^k + \bar{x}^m + a}$$

from which one can see that Eq. (3) has a unique positive equilibrium $\bar{x} = 1$.

2   Lemma and Theorem

We need the following definition.

   Definition 2.1 *A solution $\{x_n\}_{n=-3}^{\infty}$ of Eq. (3) is said to be enventually
trivial if x_n eventually equal to $\bar{x} = 1$; otherwise, the solution is said to be
nontrivial.*

Before to draw a qualitatively clear picture for the positive solutions of Eq. (3), we first establish a basic lemma which will play a key role in the proof of our main results.

Lemma 2.2 *Let $\{x_n\}_{n=-3}^{\infty}$ be a nontrivial positive solution of Eq. (3). Then the following conclusions are true for $n \geq 0$:*
(a) $(x_{n+1} - 1)(x_{n-3} - 1) > 0$
(b) $(x_{n+1} - x_{n-3})(x_{n-3} - 1) < 0$.

Proof. It follows in light of Eq. (3) that

$$x_{n+1} - 1 = \frac{(x_{n-3} - 1)[x_n x_{n-1} x_{n-2} + f(x_n, x_{n-1}, x_{n-2}, x_{n-3})x_{n-3}^k]}{x_n x_{n-1} x_{n-2} + f(x_n, x_{n-1}, x_{n-2}, x_{n-3})x_{n-3}^k + a + x_{n-3}^m},$$

$n = 0, 1, 2, \ldots$ and

$$x_{n+1} - x_{n-3} = \frac{(x_{n-3}^m + a)(1 - x_{n-3})}{x_n x_{n-1} x_{n-2} + f(x_n, x_{n-1}, x_{n-2}, x_{n-3})x_{n-3}^k + a + x_{n-3}^m},$$

$n = 0, 1, 2, \ldots$ $\qquad \square$

Remark 2.3 *If the initial conditions of the Eq. (3) satisfy $x_k \neq 1$, $k = -3, -2, -1, 0$, then, for any solution $\{x_{4n+k}\}$ of Eq. (3), $x_{4n+k} \neq 1$, $k = -3, -2, -1, 0$ for $n \geq 0$. It means that*
(a) If $x_k \neq 1$, $k = -3, -2, -1, 0$ then $x_{4n+k} \neq 1$ for $n \geq 0$.
(b) If $x_k > 1$, $k = -3, -2, -1, 0$ then $x_{4n+k} > 1$ for $n \geq 0$.
(c) If $x_k < 1$, $k = -3, -2, -1, 0$ then $x_{4n+k} < 1$ for $n \geq 0$.

Theorem 2.4 *Assume that $m, k, a \in [0, \infty)$. Then the positive equilibrium of Eq. (3) is globally asymptotically stable.*

Proof. We must prove that the positive equilibrium point $\bar{x} = 1$ of Eq. (3) is both locally asymptotically stable and globally attractive. The linearized equation of Eq. (3) about the positive equilibrium $\bar{x} = 1$ is

$$y_{n+1} = 0.y_n + 0.y_{n-1} + 0.y_{n-2} + \frac{1 + f(\bar{x}, \bar{x}, \bar{x}, \bar{x})}{2 + f(\bar{x}, \bar{x}, \bar{x}, \bar{x}) + a}y_{n-3}, \quad n = 0, 1, 2, \ldots$$

By virtue of ([11], Remark 1.3.7), $\bar{x}$ is locally asymptotically stable. It remains to verify that every positive solution $\{x_n\}_{n=-3}^{\infty}$ of Eq. (3) converges to $\bar{x} = 1$ as $n \to \infty$. Namely, we want to prove

$$\lim_{n\to\infty} x_n = \bar{x} = 1 \tag{4}$$

If the initial values of the solution satisfy $x_{-3} = x_{-2} = x_{-1} = x_0 = 1$, then Lemma 2.1 says the solution is eventually equal to 1 and, of course, (4) holds. Therefore, we assume in the following that the initial values of the solution do not satisfy $x_{-3} = x_{-2} = x_{-1} = x_0 = 1$. This means that

$$(x_{-3}, x_{-2}, x_{-1}, x_0) \neq (1, 1, 1, 1).$$

Based on the Lemma 2.1 we see that one of the following sixteen cases must occur

1) $x_{-3} \leq 1, \ x_{-2} \leq 1, \ x_{-1} \leq 1, \ x_0 \leq 1$.
2) $x_{-3} \leq 1, \ x_{-2} \leq 1, \ x_{-1} \leq 1, \ x_0 \geq 1$.
3) $x_{-3} \leq 1, \ x_{-2} \leq 1, \ x_{-1} \geq 1, \ x_0 \geq 1$.
4) $x_{-3} \leq 1, \ x_{-2} \leq 1, \ x_{-1} \geq 1, \ x_0 \leq 1$.
5) $x_{-3} \leq 1, \ x_{-2} \geq 1, \ x_{-1} \leq 1, \ x_0 \leq 1$.
6) $x_{-3} \leq 1, \ x_{-2} \geq 1, \ x_{-1} \leq 1, \ x_0 \geq 1$.
7) $x_{-3} \leq 1, \ x_{-2} \geq 1, \ x_{-1} \geq 1, \ x_0 \geq 1$.
8) $x_{-3} \leq 1, \ x_{-2} \geq 1, \ x_{-1} \geq 1, \ x_0 \leq 1$.
9) $x_{-3} \geq 1, \ x_{-2} \leq 1, \ x_{-1} \leq 1, \ x_0 \geq 1$.
10) $x_{-3} \geq 1, \ x_{-2} \leq 1, \ x_{-1} \leq 1, \ x_0 \geq 1$.
11) $x_{-3} \geq 1, \ x_{-2} \leq 1, \ x_{-1} \geq 1, \ x_0 \leq 1$.
12) $x_{-3} \geq 1, \ x_{-2} \leq 1, \ x_{-1} \geq 1, \ x_0 \leq 1$.
13) $x_{-3} \geq 1, \ x_{-2} \geq 1, \ x_{-1} \leq 1, \ x_0 \geq 1$.
14) $x_{-3} \geq 1, \ x_{-2} \geq 1, \ x_{-1} \leq 1, \ x_0 \geq 1$.
15) $x_{-3} \geq 1, \ x_{-2} \geq 1, \ x_{-1} \geq 1, \ x_0 \geq 1$.
16) $x_{-3} \geq 1, \ x_{-2} \geq 1, \ x_{-1} \geq 1, \ x_0 \leq 1$.

If case 1) occurs from the Lemma 2.1 a) we obtain $x_{4n+k} \leq 1, \ k = 0, -1, -2, -3$, for $n \geq 0$. From b) of the Lemma 2.1 we have

$$x_{4n+k} \leq x_{4(n+1)+k} \leq 1.$$

The sequences $\{x_{4n+k}\}_{n=0}^{\infty}, \ k = 0, -1, -2, -3$, are increasing with upper bound 1. So the limits $\lim\limits_{n \to \infty} x_{4n} = P, \ \lim\limits_{n \to \infty} x_{4n+1} = Q, \ \lim\limits_{n \to \infty} x_{4n+2} = R, \ \lim\limits_{n \to \infty} x_{4n+3} = L$ exist and are finite, too.

Noting that

$$x_{4n+4} = \frac{x_{4n+3}x_{4n+2}x_{4n+1}x_{4n} + f(x_{4n+3}, x_{4n+2}, x_{4n+1}, x_{4n})x_{4n}^{k+1} + x_{4n}^m + a}{x_{4n+3}x_{4n+2}x_{4n+1} + f(x_{4n+3}, x_{4n+2}, x_{4n+1}, x_{4n})x_{4n}^k + x_{4n}^m + a},$$

$$x_{4n+3} = \frac{x_{4n+2}x_{4n+1}x_{4n}x_{4n-1} + f(x_{4n+2}, x_{4n+1}, x_{4n}, x_{4n-1})x_{4n-1}^{k+1} + x_{4n-1}^m + a}{x_{4n+2}x_{4n+1}x_{4n} + f(x_{4n+2}, x_{4n+1}, x_{4n}, x_{4n-1})x_{4n-1}^k + x_{4n-1}^m + a},$$

$$x_{4n+2} = \frac{x_{4n+1}x_{4n}x_{4n-1}x_{4n-2} + f(x_{4n+1}, x_{4n}, x_{4n-1}, x_{4n-2})x_{4n-2}^{k+1} + x_{4n-2}^m + a}{x_{4n+1}x_{4n}x_{4n-1} + f(x_{4n+1}, x_{4n}, x_{4n-1}, x_{4n-2})x_{4n-2}^k + x_{4n-2}^m + a},$$

$$x_{4n+1} = \frac{x_{4n}x_{4n-1}x_{4n-2}x_{4n-3} + f(x_{4n}, x_{4n-1}, x_{4n-2}, x_{4n-3})x_{4n-3}^{k+1} + x_{4n-3}^m + a}{x_{4n}x_{4n-1}x_{4n-2} + f(x_{4n}, x_{4n-1}, x_{4n-2}, x_{4n-3})x_{4n-3}^k + x_{4n-3}^m + a},$$

$$n = 0, 1, 2, ...$$

and taking the limit on both sides of those above equalities one can see that

$$P = \frac{PQRL + f(L, R, Q, P)P^{k+1} + P^m + a}{LQR + f(L, R, Q, P)P^k + P^m + a} \implies P = 1.$$

$$L = \frac{PQRL + f(R, Q, P, L)L^{k+1} + L^m + a}{LQR + f(R, Q, P, L)L^k + L^m + a} \implies L = 1.$$

$$R = \frac{PQRL + f(Q, P, L, R)R^{k+1} + R^m + a}{LQR + f(Q, P, L, R)R^k + R^m + a} \implies R = 1.$$

$$Q = \frac{PQRL + f(P, L, R, Q)Q^{k+1} + Q^m + a}{LQR + f(P, L, R, Q)Q^k + Q^m + a} \implies Q = 1.$$

Thus, $P = Q = R = L = 1$, we have (4).
If case 2 occurs: $x_{-3} \leq 1, x_{-2} \leq 1, x_{-1} \leq 1$ and $x_0 \geq 1$. By the similar arguments, we have $\{x_{4n+1}\}_{n=0}^\infty, \{x_{4n+2}\}_{n=0}^\infty, \{x_{4n+3}\}_{n=0}^\infty$ are increasing with upper bound 1, and $\{x_{4n}\}_{n=0}^\infty$ is decreasing with lower bound 1. So, the limits

$$\lim_{n\to\infty} x_{4n} = P, \ \lim_{n\to\infty} x_{4n+1} = Q, \ \lim_{n\to\infty} x_{4n+2} = R, \ \lim_{n\to\infty} x_{4n+3} = L$$

exist and are finite, too. By the above similar method, we have

$$P = Q = R = L = 1,$$

too.
All of the rest cases we obtain the same result $\lim_{n\to\infty} x_n = \bar{x} = 1$. $\qquad\square$

3 Positive nonoscillatory solutions

In this paper we also show that the following difference equation

$$x_{n+1} = \frac{A_1}{x_n + x_{n-1}} + \frac{A_2}{x_{n-1} + x_{n-2}} + ... + \frac{A_{k-1}}{x_{n-k+2} + x_{n-k+1}} - \frac{1}{x_{n-k}}, \qquad (5)$$

$n = 0, 1, 2, ...$

where $A_1, A_2, ..., A_{k-1} \in [0, \infty)$ and

$$\sum_{i=1}^{k-1} A_i - 2 = A > 0.$$

has positive nonoscillatory solutions which converge to the positive equilibrium

$$\bar{x} = \sqrt{\frac{1}{2}\left(\sum_{i=1}^{k-1} A_i - 2\right)} = \sqrt{\frac{A}{2}}.$$

Note that the linearized equation of Eq. (5) about the positive equilibrium $\bar{x} = \sqrt{\frac{A}{2}}$ is

$$y_{n+1} = -\frac{A_1}{2A}y_n - \frac{A_1 + A_2}{2A}y_{n-1} - ... - \frac{A_{k-2} + A_{k-1}}{2A}y_{n-k+2} -$$
$$- \frac{A_{k-1}}{2A}y_{n-k+1} + \frac{4}{2A}y_{n-k}, \ n = 0, 1, 2, ...$$
$$2Ay_{n+1} + A_1 y_n + (A_1 + A_2)y_{n-1} + ... + (A_{k-2} + A_{k-1})y_{n-k+2} +$$
$$+ A_{k-1}y_{n-k+1} - 4y_{n-k} = 0 \tag{6}$$

The characteristic polynomial associated with Eq. (6) is

$$p(t) = 2At^{k+1} + A_1 t^k + (A_1 + A_2)t^{k-1} + ... + (A_{k-2} + A_{k-1})t^2 + A_{k-1}t - 4 = 0 \tag{7}$$

Since $p(0) = -4 < 0$, $p(1) = 2A + A_1 + A_1 + A_2 + A_{k-2} + A_{k-1} + A_{k-1} - 4 = 4A > 0$ with $2\bar{x}^2 = A_1 + A_2 + ... + A_{k-1} - 2 = A > 0$.

$$p'(t) = 2A(k+1)t^k + A_1 kt^{k-1} + ... + 2(A_{k-2} + A_{k-1})t + A_{k-1} > 0 \text{ for } t \in (0, 1).$$

It follows that for each $\sum_{i=1}^{k-1} A_i - 2 = A > 0$, there is a unique positive root $t_0 \in (0, 1)$ of the polynomial belonging to the interval $(0, 1)$. It means that

$$p(t_0) = 2At_0^{k+1} + A_1 t_0^k + (A_1 + A_2)t_0^{k-1} + ... +$$
$$+ (A_{k-2} + A_{k-1})t_0^2 + A_{k-1}t_0 - 4 = 0 \tag{8}$$

This fact motivated us to believe that there are solutions of Eq. (5) which have the following asymptotics

$$x_n = \bar{x} + at_0^n + o(t_0^n) \tag{9}$$

where $a \in \mathbb{R}$ and t_0 is the above mentioned root of Eq. (7). We solve the open problem, showing that such a solution exists, developing Berg's ideas

in [5] which are based on the asymptotics. The asymptotics for solutions of difference equation have been investigated by L. Berg and S. Stević, see, for example [4-8], [10-12, 27-29, 31, 32, 34-39] and the reference therein. The problem is solved by constructing appropriate sequences y_n and z_n with

$$y_n \leq x_n \leq z_n \tag{10}$$

for sufficiently large n. In [5, 6] some methods can be found for the construction of these bounds, see, also [7, 8].

From (9) we expect that for $k \geq 2$ such solutions have the first three members in their asymptotics in the following form

$$\varphi_n = \bar{x} + at_0^n + bt_0^{2n} \tag{11}$$

This is proved by developing Berg's ideas in [5] which are based on asymptotics. We need the following result in the proof of main theorem. The proof of the following theorem can be found in [38, 39].

Theorem 3.1 *Let $f : I^{k+2} \to I$ be a continuous and nondecreasing function in each argument on the interval $I \subset \mathbb{R}$, and let $\{y_n\}$ and $\{z_n\}$ be sequences with $y_n \leq z_n$ for $n \geq n_0$ and such that*

$$y_{n-k} \leq f(n, y_{n-k+1}, ..., y_{n+1}), \ f(n, z_{n-k+1}, ..., z_{n+1}) \leq z_{n-k} \ \ for \ \ n > n_0 + k - 1 \tag{12}$$

Then there is a solution of the following difference equation

$$x_{n-k} = f(n, x_{n-k+1}, ..., x_{n+1}) \tag{13}$$

with property (10) for $n \geq n_0$.

Theorem 3.2 *For each $\sum_{i=1}^{k-1} A_i - 2 = A > 0$ and $A_1, A_2, ..., A_{k-1} \in [0, \infty)$ there is a nonoscillatory solution of Eq. (5) converging to the positive equilibrium $\bar{x} = \sqrt{\dfrac{A}{2}}$, as $n \to \infty$.*

Proof First, Eq. (5) can be written in the form

$$x_{n-k} = \left(\frac{A_1}{x_n + x_{n-1}} + \frac{A_2}{x_{n-1} + x_{n-2}} + ... + \frac{A_{k-1}}{x_{n-k+2} + x_{n-k+1}} - x_{n+1} \right)^{-1}.$$

Setting

$$F(x_{n-k}, x_{n-k+1}, ..., x_n, x_{n+1}) = \tag{14}$$

$$= \left(\frac{A_1}{x_n + x_{n-1}} + \frac{A_2}{x_{n-1} + x_{n-2}} + ... + \frac{A_{k-1}}{x_{n-k+2} + x_{n-k+1}} - x_{n+1} \right)^{-1} - x_{n-k}$$

We expect the solutions of Eq. (5) have the asymptotic appropriation (11).

$$F = F(\varphi_{n-k}, \varphi_{n-k+1}, ..., \varphi_n, \varphi_{n+1}) =$$

$$= \left(\frac{A_1}{2\bar{x} + at^n + bt^{2n} + at^{n-1} + bt^{2n-2}} + \frac{A_2}{2\bar{x} + at^{n-1} + bt^{2n-2} + at^{n-2} + bt^{2n-4}} + ... \right.$$

$$\left. + \frac{A_{k-1}}{2\bar{x} + at^{n-k+1} + bt^{2n-2k+2} + at^{n-k-2} + bt^{2n-2k+4}} - \bar{x} - at^{n+1} - bt^{2n+2} \right)^{-1} -$$

$$- \bar{x} - at^{n-k} - bt^{2n-2k}.$$

$$F = \frac{a}{4t^k}[2At^{k+1} + A_1 t^k + (A_1 + A_2)t^{k-1} + ... + (A_{k-2} + A_{k-1})t^2 + A_{k-1}t - 4]t^n +$$

$$+ \frac{b}{4t^{2k}}[2At^{2k+2} + A_1 t^{2k} + (A_1 + A_2)t^{2k-2} + ... + (A_{k-2} + A_{k-1})t^4 + A_{k-1}t^2 - 4]t^{2n} +$$

$$+ \frac{a^2}{16\bar{x}} \left(A_1 + A_1 t^{-1} + A_2 t^{-1} + A_2 t^{-2} + ... + A_{k-1}t^{-k+1} + A_{k-1}t^{-k+2} + 2At \right)^2 + o(t^{2n})$$

$$F = \frac{ap(t)}{4t^k}t^n + \left\{ \frac{bp(t^2)}{4t^{2k}} + \frac{a^2}{16\bar{x}} \left[(A_1 + A_1 t^{-1} + A_2 t^{-1} + A_2 t^{-2} + ... + A_{k-1}t^{-k+1} + \right. \right.$$

$$(15)$$

$$+ A_{k-1}t^{-k+2} + 2At)^2 - 2(A_1 + A_1 t^{-2} + 2A_1 t^{-1} + ... + A_{k-1}t^{-2k+2} + A_{k-1}t^{-2k+4} +$$

$$\left. \left. + 2A_{k-1}t^{-2k+3}) \right] \right\} t^{2n} + o(t^{2n}).$$

As mentioned earlier exists $t_0 \in (0,1)$ such that $p(t_0) = 0$ and $0 < t_0^2 < t_0 < 1$, $p(t_0^2) < 0$. Posing in (15), $t = t_0$ we obtain

$$F = \left\{ \frac{bp(t_0^2)}{4t_0^{2k}} + \frac{a^2}{16\bar{x}} \left[(A_1 + A_1 t_0^{-1} + A_2 t_0^{-1} + A_2 t_0^{-2} + ... \right. \right.$$

$$+ A_{k-1}t_0^{-k+1} + A_{k-1}t_0^{-k+2} + 2At_0)^2 - 2(A_1 + A_1 t_0^{-2} + 2A_1 t_0^{-1} + ... +$$

$$\left. \left. + A_{k-1}t_0^{-2k+2} + A_{k-1}t_0^{-2k+4} + 2A_{k-1}t_0^{-2k+3}) \right] \right\} t_0^{2n} + o(t_0^{2n}).$$

Posing

$$B = \frac{p(t_0^2)}{4t_0^{2k}} < 0$$

$$C = \frac{a^2}{16\bar{x}} \left[(A_1 + A_1 t_0^{-1} + A_2 t_0^{-1} + A_2 t_0^{-2} + ... + A_{k-1}t_0^{-k+1} + A_{k-1}t_0^{-k+2} + 2At_0)^2 - \right.$$

$$\left. - 2(A_1 + A_1 t_0^{-2} + 2A_1 t_0^{-1} + ... + A_{k-1}t_0^{-2k+2} + A_{k-1}t_0^{-2k+4} + 2A_{k-1}t_0^{-2k+3}) \right].$$

Therefore, we have $F = (Bb + C)t_0^{2n} + o(t_0^{2n})$.
Setting

$$H_{t_0}(q) = Bq + C = 0 \Longrightarrow q_0 = -\frac{C}{B}, \ H'_{t_0}(q) = B < 0.$$

We obtain that there are $q_1 < q_0$ and $q_2 > q_0$ such that

$$H_{t_0}(q_1) > 0, H_{t_0}(q_2) < 0, q_1 < q_0 < q_2.$$

We assume that $a \neq 0$. If $\hat{\varphi}_n = \overline{x} + at_0^n + q_0 t_0^{2n}$, we obtain

$$F(\hat{\varphi}_{n-k}, \hat{\varphi}_{n-k+1}, ..., \hat{\varphi}_n, \hat{\varphi}_{n+1}) \underset{n \to \infty}{\sim} \left\{ \frac{q_0 p(t_0^2)}{4t_0^{2k}} + C \right\} t_0^{2n} + o(t_0^{2n}).$$

With the notations

$$y_n = \overline{x} + at_0^n + q_1 t_0^{2n}, \ \ z_n = \overline{x} + at_0^n + q_2 t_0^{2n}.$$

We get

$$F(y_{n-k}, y_{n-k+1}, ..., y_n, y_{n+1}) \sim (q_1 B + C)t_0^{2n},$$
$$F(z_{n-k}, z_{n-k+1}, ..., z_n, z_{n+1}) \sim (q_2 B + C)t_0^{2n}.$$

These relations show that inequalities (12) are satisfied for sufficiently large n, where $f = F + x_{n-k}$ and F is given by (14). Because the function

$$f(x_{n-k+1}, x_{n-k+2}, ..., x_n, x_{n+1})$$

is continuous and nondecreasing on the interval $[\overline{x}, +\infty)$. We easily have $f(\overline{x}, \overline{x}, ..., \overline{x}, \overline{x}) = \overline{x}$. We can apply Theorem 4.1 with $I = [\overline{x}, +\infty)$ and see that there is an $n_0 \geq 0$ and a solution of Eq. (5) with the asymptotics $x_n = \hat{\varphi}_n + o(t_0^{2n})$ for $n \geq n_0$, where $b = q_0$ in $\hat{\varphi}_n$. In particular, the solution converge monotonically to the positive equilibrium $\overline{x} = \sqrt{\frac{A}{2}}$, for $n \geq n_0$. Hence, the solution x_{n+n_0+k} is also such a solution when $n \geq -k$. $\qquad\square$

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