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Relative Ritt Order of Entire Dirichlet Series

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Abstract

We introduce the concept of relative Ritt order of entire Dirichlet series and prove sum and product theorems. We also show that an entire Dirichlet series and its derivative have the same relative Ritt order (finite) under certain conditions.

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1 Introduction, Definition and Lemmas

For entire functions g_1 and g_2 let

$$G_1(r) = max\{|g_1(z)| : |z| = r\} \text{ and } G_2(r) = max\{|g_2(z)| : |z| = r\}.$$

If g_1 is non-constant then $G_1(r)$ is strictly increasing and a continuous function of r and its inverse

$$G_1^{-1} : (|g_1(0)|, \infty) \to (0, \infty)$$

exists and $\lim_{R\to\infty} G_1^{-1}(R) = \infty$.

Bernal [4] introduced the definition of relative order of g_1 with respect to g_2, denoted by $\rho_{g_2}(g_1)$, as follows:

$$\rho_{g_2}(g_1) = inf\{\mu > 0 : G_1(r) < G_2(r^{\mu}) \text{ for all } r > r_0(\mu) > 0\}.$$

After Bernal, several papers on relative order of entire functions have appeared in the literature where growing interest of workers on this topic has been noticed {see for example [1], [2], [3], [11], [12], [13], [14], [15]}.

Before we pass on, we remark that the papers {[7], [9], [10]} contains investigations on relative order (H:K) of entire functions, but Bernal's analysis including subsequent studies after Bernal have little relevance to the studies made in the above papers.

During the past decades, several authors {see for example [16],[17],[19] } made close investigations on the properties of entire Dirichlet series related to Ritt order. At this stage it therefore seems reasonable to define suitably the relative Ritt order of entire Dirichlet series with respect to an entire function and to enquire its basic properties in the new context. Proving some preliminary theorems on the relative Ritt order, we obtain sum and product theorems and we show that the relative Ritt order (finite) of an entire function represented by Dirichlet series is the same as its derivative, under some restrictions.

Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ defined by everywhere absolutely convergent Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{s\lambda_n} \qquad ...(1)$$

where $0 < \lambda_n < \lambda_{n+1}$ $(n \geq 1), \lambda_n \to \infty$ as $n \to \infty$ and $a_n^{,}s$ are complex constants.

If σ_c and σ_a denote respectively the abscissa of convergence and absolute convergence of (1) then in this case clearly $\sigma_c = \sigma_a = \infty$.

Let $F(\sigma) = {}_{-\infty<t<\infty}^{l.u.b.} \; |f(\sigma + it)|$. ...(2)

Then the Ritt order [16] of $f(s)$, denoted by $\rho(f)$ is given by

$$\rho(f) = \limsup_{\sigma\to\infty} \frac{loglogF(\sigma)}{\sigma}.$$

In other words $\rho(f) = inf\{\mu > 0 : logF(\sigma) < exp(\sigma\mu)$ for all $\sigma > R(\mu)\}$.

The following definition is now introduced.

Definition 1. The relative Ritt order of $f(s)$ with respect to an entire $g(s)$ is defined by

$$\rho_g(f) = inf\{\mu > 0 : logF(\sigma) < G(\sigma\mu) \text{ for all large } \sigma\}$$

where $G(r) = max\{|g(s)| : |s| = r\}$.

Clearly $\rho_g(f) = \rho(f)$ if $g(s) = e^s$.

The following analogous definition from [4] will be needed.

Definition 2. A nonconstant entire function $g(s)$ is said to have the property (A) if for any $\delta > 1$ and positive $\sigma, [G(\sigma)]^2 \leq G(\sigma^\delta)$ holds where $G(\sigma) =$max $\{|g(s)| : |s| = \sigma\}$.

Property (A) has been closely studied in [4] where several examples may be found holding the property (A) as well do not hold the property (A).

The symbols f, f_1, f_2 etc. will be reserved for nonconstant entire functions each represented by everywhere absolutely convergent Dirichlet series of the form (1) and g, g_1, g_2 etc. will stand for nonconstant entire functions. The symbols $G(r), G_1(r), G_2(r)$ etc. for the functions g, g_1, g_2 etc. will have the same meaning as in the beginning of this section and $F(\sigma), F_1(\sigma), F_2(\sigma)$ etc. for the functions f, f_1, f_2 etc. will have analogous meaning as in (2).

The following known lemmas will be required in the sequel.

Lemma 1[4]. *If $\alpha > 1, 0 < \beta < \alpha$ then $G(\alpha r) > \beta G(r)$ for all large r.*

Lemma 2[4]. *If g is transcendental then*
$$\lim_{r\to\infty} \frac{G(r^k)}{r^n G(r)} = \infty$$
where $k > 1$ and n is any positive integer.

2 Preliminary Theorem

In the notations of section 1, we have the following theorem.

Theorem 1. *(a)* $\rho_g(f) = \limsup_{\sigma\to\infty} \frac{G^{-1} log F(\sigma)}{\sigma}$.

(b) If f(s) is a Dirichlet polynomial and g(s) is transcendental then $\rho_g(f) = 0$.

(c) If f(s) is a Dirichlet polynomial and g(s) is a polynomial of degree not less than two than $\rho_g(f) = 0$.

(d) If $F_1(\sigma) \le F_2(\sigma)$ for all large σ, then $\rho_g(f_1) \le \rho_g(f_2)$.

(e) If $G_1(\sigma) \le G_2(\sigma)$ for all large σ, then $\rho_{g_1}(f) \ge \rho_{g_2}(f)$.

Proof :(a) This follows from definition.

(b) Let f be of the form $f(s) = \sum_{k=1}^{n} a_k e^{s\lambda_k}$. Then

$$\begin{aligned} F(\sigma) &= \underset{-\infty<t<\infty}{l.u.b.} \left|\sum_{k=1}^{n} a_k e^{(\sigma+it)\lambda_k}\right| \\ &\le \underset{-\infty<t<\infty}{l.u.b.} \left[\sum_{k=1}^{n} |a_k| e^{\sigma\lambda_k}\right] \\ &\le n.\max_{k=1,2,\ldots,n} |a_k| e^{\sigma\lambda_n}, \end{aligned}$$
since we may clearly assume that σ is positive

$= M e^{\sigma\lambda_n}$, say where $M = n.\max_{k=1,2,\ldots,n} |a_k|$ is a constant. . .(3)

On the other hand, since $g(s)$ is transcendental, $G(\sigma) > K\sigma^m$ for all large σ and $m > 0$, where K is a constant large at pleasure. We have then for $\mu > 0$

$$\begin{aligned} G(\sigma\mu) &> K\sigma^m\mu^m \\ &> log M + \sigma\lambda_n \text{ , for all large } \sigma \text{ and a fixed } m > 1 \\ &\ge log F(\sigma) \text{ from (3).} \end{aligned}$$

So for all large σ and arbitrary $\mu > 0$, $\frac{G^{-1}logF(\sigma)}{\sigma} < \mu$ and this gives that $\rho_g(f) = 0$.

(c) Let $g(s) = b_0 s^m + b_1 s^{m-1} + ... + b_m, b_0 \neq 0, m \geq 2$ and $f(s) = \sum_{k=1}^{n} a_k e^{s\lambda_k}$. Since σ clearly can be taken as positive, the inequality $g(\sigma) > \frac{1}{2}|b_0|\sigma^m$ may be derived for all large σ. Also, as in *(b)* $F(\sigma) \leq Me^{\sigma\lambda_n}$ for all large σ. From the first inequality, we have for all large σ

$$G^{-1}[\tfrac{1}{2}|b_0|\sigma^m] < \sigma.$$

Taking $R = \frac{1}{2}|b_0|\sigma^m$, so that $\sigma = (\frac{2R}{|b_0|})^{\frac{1}{m}}$,we obtain from above

$$G^{-1}(R) < (\tfrac{2R}{|b_0|})^{\frac{1}{m}} \text{ for all large } R.$$

We can now replace R by $logF(\sigma)$, which is clearly justifiable because $logF(\sigma)$ is a continuous strictly increasing function of σ tending to ∞ { see [6], [18] },and ultimately obtain

$$G^{-1}[logF(\sigma)] < (\tfrac{2logF(\sigma)}{|b_0|})^{\frac{1}{m}} \leq (\tfrac{2(logM+\lambda_n\sigma)}{|b_0|})^{\frac{1}{m}}.$$

Therefore
$$\rho_g(f) = \limsup_{\sigma\to\infty} \frac{G^{-1}(logF(\sigma))}{\sigma}$$
$$\leq \lim_{\sigma\to\infty} (2.\frac{logM+\lambda_n\sigma)}{|b_0|\sigma^m})^{\frac{1}{m}}$$
$$= 0, \text{ since } m \geq 2.$$

(d) For arbitrary $\epsilon > 0$ and for all large σ, we can write from *(a)*

$$F_2(\sigma) < exp[G\{\sigma(\rho_g(f_2) + \epsilon)\}].$$

Since $F_1(\sigma) \leq F_2(\sigma)$ for all large σ , we obtain

$$\rho_g(f_1) = \limsup_{\sigma\to\infty} \frac{G^{-1}logF_1(\sigma)}{\sigma} \leq \rho_g(f_2) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this proves *(d)*.

(e) We have $\sigma \leq G_1^{-1}(G_2(\sigma))$. Writing $\mu = G_2(\sigma)$, we obtain $G_2^{-1}(\mu) \leq G_1^{-1}(\mu)$ and thus

$$\rho_{g_2}(f) = \limsup_{\sigma\to\infty} \frac{G_2^{-1}logF(\sigma)}{\sigma} \leq \limsup_{\sigma\to\infty} \frac{G_1^{-1}logF(\sigma)}{\sigma}$$
$$= \rho_{g_1}(f).$$

3 Sum and Product Theorems

In this section we assume that f_1, f_2 etc. are entire functions of s defined by everywhere absolutely convergent *ordinary* Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}, \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ etc. The product of two such series is considered by Dirichlet product method, which is also everywhere absolutely convergent { see [8], pp 66 }.

Theorem 2. *Let $g(s)$ be an entire function having the property (A). Then*
(i) $\rho_g(f_1 \pm f_2) \leq max\{\rho_g(f_1), \rho_g(f_2)\}$, sign of equality holds when $\rho_g(f_1) \neq \rho_g(f_2)$
and (ii)$\rho_g(f_1 f_2) \leq max\{\rho_g(f_1), \rho_g(f_2)\}$.

Proof :(i) We may suppose that $\rho_g(f_1)$ and $\rho_g(f_2)$ both are finite, because in the contrary case the inequality follows immediately. We prove *(i)* for addition only, because the proof for subtraction is analogous.

Let $f = f_1 + f_2, \rho = \rho_g(f), \rho_i = \rho_g(f_i), i = 1,2$ and $\rho_1 \le \rho_2$.

For arbitrary $\epsilon > 0$ and for all large σ, we have from Theorem 1(a)

$F_1(\sigma) < exp[G(\sigma(\rho_1+\epsilon))] \le exp[G(\sigma(\rho_2+\epsilon))]$

and $F_2(\sigma) < exp[G(\sigma(\rho_2+\epsilon))]$.

So for all large σ

$$\begin{aligned} F(\sigma) &\le F_1(\sigma) + F_2(\sigma) \\ &\le 2exp[G(\sigma(\rho_2+\epsilon))] \\ &< exp[G(\sigma(\rho_2+\epsilon))]^2, \text{ since for all } x, 2\ exp(x) < exp(x^2) \\ &\le exp[G\{\sigma(\rho_2+\epsilon)\}^\delta] \text{ for every } \delta > 1, \text{ by property (A).} \end{aligned}$$

Therefore $\frac{G^{-1}logF(\sigma)}{\sigma} < (\rho_2+\epsilon)^\delta \sigma^{\delta-1}$, for all large σ.

Taking first $\delta \to 1+0$ and then limit superior as $\sigma \to \infty$ and noting that $\epsilon > 0$ is arbitrary, we obtain $\rho \le \rho_2$. This proves the first part of *(i)*.

For the second part of *(i)*, let $\rho_1 < \rho_2$ and suppose that $\rho_1 < \mu < \lambda < \rho_2$. Then for all large σ

$$F_1(\sigma) < exp[G(\sigma\mu)] \quad \quad ...(4)$$

and there exists an increasing sequence $\{\sigma_n\}, \sigma_n \to \infty$ such that

$$F_2(\sigma_n) > exp[G(\sigma_n\lambda)] \quad \quad ...(5)$$

for $n = 1,2,3,...$.

Using Lemma 1, by setting $\alpha = \frac{\lambda}{\mu}, r = \sigma\mu, \beta = 1+\epsilon, 0 < \epsilon < 1$ such that $1 < \beta < \alpha$, we obtain

$$G(\tfrac{\lambda}{\mu}\sigma\mu) > (1+\epsilon)G(\sigma\mu), \text{i.e. } G(\lambda\sigma) > (1+\epsilon)G(\sigma\mu).$$

Therefore using (5) and then (4) and the fact that $G(\sigma) > \frac{log2}{\epsilon}$ for all large σ, we obtain

$$\begin{aligned} F_2(\sigma_n) &> exp[G(\sigma_n\lambda)] \\ &> exp[(1+\epsilon)G(\sigma_n\mu)] \\ &> 2exp[G(\sigma_n\mu)], \text{ for all large } n \\ &> 2F_1(\sigma_n), \text{ for all large } n. \end{aligned} \quad \quad ...(6)$$

$$\begin{aligned} \text{Now } F(\sigma_n) &\ge F_2(\sigma_n) - F_1(\sigma_n) \\ &> F_2(\sigma_n) - \tfrac{1}{2}F_2(\sigma_n) \text{ , using (6)} \\ &= \tfrac{1}{2}F_2(\sigma_n) \\ &> \tfrac{1}{2}exp[G(\sigma_n\lambda)] \text{ , from (5)} \\ &> exp[(1-\epsilon)G(\sigma_n\lambda)] \text{ , for all large } n. \end{aligned}$$

Let $\rho_1 < \lambda_1 < \lambda < \rho_2$ and $0 < \epsilon < \frac{\lambda-\lambda_1}{\lambda}$ (which is clearly permissible). Using Lemma 1, by setting $\alpha = \frac{\lambda}{\lambda_1}, \beta = \frac{1}{1-\epsilon}, r = \sigma\lambda_1$, we have, because $0 < \beta < \alpha$

$G(\frac{\lambda}{\lambda_1}\sigma\lambda_1) > \frac{1}{1-\epsilon}G(\sigma\lambda_1)$,

i.e. $(1-\epsilon)G(\sigma\lambda) > G(\sigma\lambda_1)$.

Hence for all large n , $F(\sigma_n) > exp[G(\sigma_n\lambda_1)]$,

i.e. $\frac{G^{-1}logF(\sigma_n)}{\sigma_n} > \lambda_1$ for all large n.

This gives $\rho \geq \lambda_1$. Since λ and λ_1 both are arbitrary in the interval (ρ_1, ρ_2), we have $\rho \geq \rho_2 = \max\{\rho_1, \rho_2\}$,
i.e. $\rho(f_1 + f_2) \geq \max\{\rho_g(f_1), \rho_g(f_2)\}$.
This in conjunction with the first part of *(i)* gives

$$\rho(f_1 + f_2) = \max\{\rho_g(f_1), \rho_g(f_2)\}$$

which proves *(i)* completely.

(ii)Let $f = f_1 f_2$ and the notations ρ, ρ_1 and ρ_2 have the analogous meanings as in *(i)*. If $\rho_1 \leq \rho_2$ then for arbitrary $\epsilon > 0$ and for all large σ

$$\begin{aligned} F(\sigma) &\leq F_1(\sigma)F_2(\sigma) \\ &< exp[G(\sigma(\rho_1 + \epsilon))].exp[G(\sigma(\rho_2 + \epsilon))] \\ &\leq exp[2G(\sigma(\rho_2 + \epsilon))] \\ &\leq exp[G(\sigma(\rho_2 + \epsilon))]^2 \\ &\leq exp[G(\sigma(\rho_2 + \epsilon))^\delta] \end{aligned}$$

for every $\delta > 1$, by property (A).

The above gives $\frac{G^{-1}logF(\sigma)}{\sigma} \leq (\rho_2 + \epsilon)^\delta \sigma^{\delta - 1}$ for all large σ. Letting $\delta \to 1+0$ and then considering the fact that $\epsilon > 0$ is arbitrary, we obtain $\rho \leq \rho_2$ which proves the theorem.

4 Relative Ritt order of the derivative

Theorem 3. *Let $f(s)$ be an entire function defined by the Dirichlet series (1) having finite Ritt order $\rho(f)$ and $f'(s)$ be its derivative. Then $\rho_g(f) = \rho_g(f')$ where $g(s)$ is a transcendental entire function.*

Proof : It is known {[16], p 139} that for all large values of σ and arbitrary $\epsilon > 0$

$$F(\sigma) - \epsilon < (\sigma - \sigma_0)F'(\sigma) + |f(s_0)| \qquad \ldots (7)$$

where $s_0 = \sigma_0 + it_0$ is a fixed complex number and $F'(\sigma) =_{-\infty<t<\infty}^{l.u.b.} |f'(\sigma + it)|$. The inequality (7) implies

$$F(\sigma) < \sigma F'(\sigma) + A + \epsilon$$

where A is a constant. Taking logarithm, we see that for all large values of σ

$$\begin{aligned} logF(\sigma) &< log(\sigma F'(\sigma)) + B_\sigma \text{ where } B_\sigma \to 0 \text{ as } \sigma \to \infty \\ &< logF'(\sigma) + log\sigma + B_\sigma \\ &< logF'(\sigma) + \sigma(\rho_g(f') + \epsilon) + B_\sigma \\ &< logF'(\sigma) + \sigma(\rho_g(f') + 2\epsilon) \\ &< G[\sigma(\rho_g(f') + \epsilon)] + \sigma(\rho_g(f') + 2\epsilon) \\ &< G[\sigma(\rho_g(f') + 2\epsilon)] \qquad \ldots (8) \end{aligned}$$

because $\frac{G[\sigma(\rho_g(f')+\epsilon)]+\sigma(\rho_g(f')+2\epsilon)}{G[\sigma(\rho_g(f')+2\epsilon)]} < 1$ for all large σ on using {[4],(d) , p 213} and {[5], p165}.
From (8) $\rho_g(f) =_{\sigma\to\infty}^{limsup} \frac{G^{-1}logF(\sigma)}{\sigma} \leq \rho_g(f') + 2\epsilon$.

Since $\epsilon > 0$ is arbitrary , $\rho_g(f) \leq \rho_g(f')$.

To obtain the reverse inequality, we use the following inequality from {[16], p139}

$$F'(\sigma) - \epsilon \leq \frac{1}{\delta}F(\sigma + \delta) \quad \ldots (9)$$

where $\epsilon > 0$ is arbitrary and $\delta > 0$ is fixed.

So
$$\begin{aligned} logF'(\sigma) &\leq log(\tfrac{1}{\delta}F(\sigma + \delta) + \epsilon) \\ &= logF(\sigma + \delta) + log(\tfrac{1}{\delta} + \tfrac{\epsilon}{F(\sigma+\delta)}) \\ &\leq G[(\sigma + \delta)(\rho_g(f) + \epsilon)] + log(\tfrac{1}{\delta} + \tfrac{\epsilon}{F(\sigma+\delta)}) \\ &\leq G[(\sigma + \delta)(\rho_g(f) + 2\epsilon)] \text{ for all large } \sigma. \end{aligned}$$

Therefore
$$\begin{aligned} \rho_g(f') &= \limsup_{\sigma\to\infty} \frac{G^{-1}logF'(\sigma)}{\sigma} \\ &\leq \rho_g(f) + 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\rho_g(f') \leq \rho_g(f)$ which proves the theorem.

If we assume $g(0) = 0$, a simpler proof of the following theorem may be provided which relates the Ritt order of f relative to g and to its derivative g'.

Theorem 4. *Let $f(s)$ be an entire function defined by the Dirichlet series (1) and $g(s)$ be an entire transcendental function with $g(0) = 0$, then*

$$\frac{1}{2}\rho_g(f) \leq \rho_{g'}(f) \leq \rho_g(f).$$

The following lemma is required.

Lemma 3. *If g is transcendental with $g(0) = 0$ then for all large r and $0 < \delta < 1$*

$$G(r^\delta) < \bar{G}(r) < G(2r)$$

where $\bar{G}(r) = max\ \{|g'(z)| : |z| = r\}$.

Proof : We may write

$$g(z) = \int_0^z g'(t)dt$$

where the line of integration is the segment from $z = 0$ to $z = re^{i\theta_0}, r > 0$. Let $z_1 = re^{i\theta_1}$ be a point such that $|g(z_1)| = \max\{|g(z)| : |z| = r\}$.

We then obtain

$$G(r) = |g(z_1)| = |\int_0^{z_1} g'(t)dt| \leq r\bar{G}(r). \quad \ldots (10)$$

Let C denote the circle $|t - z_0| = r$ where $z_0, |z_0| = r$ is such that $|g'(z_0)| = \max\ \{|g'(z)| : |z| = r\}$. Then

$$\begin{aligned} \bar{G}(r) = |g'(z_0)| &= |\frac{1}{2\pi i}\int_C \frac{g(t)}{(t-z_0)^2}dt| \\ &\leq \frac{1}{2\pi}.\frac{G(2r)}{r^2}.2\pi r = \frac{G(2r)}{r}. \quad \ldots (11) \end{aligned}$$

From (10) and (11) we obtain

$$\frac{G(r)}{r} \leq \bar{G}(r) \leq \frac{G(2r)}{r}. \quad \ldots (12)$$

From Lemma 2 it follows that $G(r^k) > r^nG(r)$ for all large r and for every positive integer n where $k > 1$. Replacing r by r^δ and assuming $k = \frac{1}{\delta}$, this implies

$$G(r^{k\delta}) > r^{n\delta}G(r^\delta) \geq rG(r^\delta)$$

where n is large enough to ensure $n\delta \geq 1$ and thus

$$G(r) > rG(r^{\delta}).$$

Using (12) for all large r

$$G(r^{\delta}) < \frac{G(r)}{r} \leq \bar{G}(r) \leq \frac{G(2r)}{r}.$$

Hence $G(r^{\delta}) < \bar{G}(r) < G(2r)$ for all large r where $0 < \delta < 1$.

This proves the lemma.

Proof of Theorem 4 : Since $g(s)$ is transcendental and $g(0) = 0$, we have by Lemma 3 for all large σ and $0 < \delta < 1$

$$G(\sigma^{\delta}) < \bar{G}(\sigma) < G(2\sigma)$$

where $\bar{G}(\sigma) = \max\{|g'(s)| : |s| = \sigma\}$.

By computations it follows that

$$\frac{1}{2}G^{-1}(\sigma) < \bar{G}^{-1}(\sigma) < [G^{-1}(\sigma)]^{\frac{1}{\delta}}$$

for all large σ. Therefore we can write for all large σ

$$\frac{1}{2}\frac{G^{-1}[logF(\sigma)]}{\sigma} < \frac{\bar{G}^{-1}[logF(\sigma)]}{\sigma} < \frac{\{G^{-1}[logF(\sigma)]\}^{\frac{1}{\delta}}}{\sigma},$$ since $logF(\sigma)$ is increasing and tending to infinity as $\sigma \to \infty$ { see [6] , [18] }.

Letting $\delta \to 1 - 0$, we obtain for all large σ

$$\frac{1}{2}\frac{G^{-1}logF(\sigma)}{\sigma} < \frac{\bar{G}^{-1}logF(\sigma)}{\sigma} \leq \frac{G^{-1}logF(\sigma)}{\sigma}$$

and this gives

$$\frac{1}{2}\rho_g(f) \leq \rho_{g'}(f) \leq \rho_g(f)$$

which proves the theorem.

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