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Variational Iteration Method for a Class of Nonlinear Differential Equations

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Abstract

In this paper, we present the approximate analytic solutions of a large class of nonlinear differential equations with variable coefficients by using variational iteration method (VIM). Some numerical examples are selected to illustrate the effectiveness and simplicity of the method.

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1 Introduction

The VIM was developed by He in [1]-[3]. In recent years a great deal of attention has been devoted to the study of the method. The reliability of the method and the reduction in the size of the computational domain give this method a wide applicability.

The VIM based on the use of restricted variations and correction functionals which has found a wide application for the solution of nonlinear ordinary and partial differential equations, e.g., [4]-[9]. This method does not require the presence of small parameters in the differential equation, and provides the solution (or an approximation to it) as a sequence of iterates. The method does not require that the nonlinearities be differentiable with respect to the dependent variable and its derivatives.

The aim of this paper is to extend the VIM to find the approximate analytic solutions of the following second order nonlinear ODE with variable coefficients

$$u''(t) + \frac{h'(t)}{h(t)}u'(t) + f\left(t, u(t)\right) = g(t), \quad u(0) = A, \quad u'(0) = B \tag{1}$$

where $f\left(t, u(t)\right)$ and $g(t)$ are continuous real valued functions, $h(t)$ is a continuous and differentiable function with $h(t) \neq 0$. Approximate solutions to the above problem were presented in [10] by applying the Adomian decomposition method.

The well known physical equations such as Bratu, Emden-Fowler, Lane-Emden, Poisson-Boltzmann, Lagerstrom, etc. are special cases of the above equation.

2 Variational Iteration Method

Now, to illustrate the basic concept of the method, we consider the following general nonlinear differential equation given in the form

$$Lu(t) + Nu(t) = g(t)$$

where L is a linear operator, N is a nonlinear operator and $g(t)$ is a known analytical function. We can construct a correction functional according to the variational method as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,s)\left(Lu_n(s) + N\widetilde{u}_n(s) - g(s)\right)ds, \quad n \geq 0$$

where λ is a general Lagrange multiplier, which can be identified optimally via variational theory, u_n is the n^{th} approximate solution and $\widetilde{u}_n$ denotes a restricted variation, which means $\delta\widetilde{u}_n = 0$. Successive approximations, $u_{n+1}(t)$, will be obtained by applying the obtained Lagrange multiplier and a properly chosen initial approximation $u_0(t)$. Consequently, the solution is given by $u = \lim_{n\to\infty} u_n$. For error estimates and convergence of VIM, see [11].

3 Implementation of the Method

In this section, for solving equation (1) by means of VIM, we construct the correction functional as follows :

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,s)\left(u_n''(s) + \frac{h'(s)}{h(s)}u_n'(s) + \widetilde{f}\left(s, u_n(s)\right) - g(s)\right)ds.$$

Making the above correction functional stationary with respect to u_n, noticing that $\delta u_n(0) = 0$, yields

$$\begin{aligned} \delta u_{n+1}(t) &= \delta u_n(t) \\ &\quad + \delta \int_0^t \lambda(t,s) \left(u_n''(s) + \frac{h'(s)}{h(s)} u_n'(t) + \widetilde{f}\,(s, u_n(s)) - g(s) \right) ds. \\ &= \delta u_n(t) + \left(\lambda(t,s) \frac{h'(s)}{h(s)} \delta u_n(s) + \lambda(t,s) \delta u_n'(s) - \frac{\partial \lambda(t,s)}{\partial s} \delta u_n(s) \right) \Bigg|_{s=t} \\ &\quad + \int_0^t \left[\left(\frac{\partial^2 \lambda(t,s)}{\partial s^2} - \frac{\partial}{\partial s} \left[\lambda(t,s) \frac{h'(s)}{h(s)} \right] \right) \delta u_n(s) \right] ds \\ &= 0. \end{aligned}$$

So, the following stationary conditions are obtained :

$$\frac{\partial^2 \lambda(t,s)}{\partial s^2} - \frac{\partial}{\partial s} \left[\lambda(t,s) \frac{h'(s)}{h(s)} \right] = 0$$

$$1 + \lambda(t,t) \frac{h'(t)}{h(t)} - \frac{\partial \lambda(t,s)}{\partial s} \Bigg|_{s=t} = 0, \quad \lambda(t,t) = 0.$$

Therefore, the Lagrange multiplier can be readily identified

$$\lambda(t,s) = h(s) \left(\int \frac{ds}{h(s)} - \int \frac{dt}{h(t)} \right).$$

Consequently, the iteration formula can be obtained as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,s) \left(u_n''(s) + \frac{h'(s)}{h(s)} u_n'(s) + f(s, u_n(s)) - g(s) \right) ds.$$

4 Some Examples

In this section, we applied the proposed method of VIM to several equations of type (1) as in the following examples. The first one is an original example for equation (1), the other three examples are well known IVP's of mathematical physics which solved several methods before.

Example 4.1 *Consider the nonlinear equation*

$$u''(t) - \frac{2+t}{1+t} u'(t) + u^2(t) = t^2 e^{2t}$$

with initial conditions $u(0) = 0, u'(0) = 1$. The exact solution of this problem is $u(t) = te^t$. Since $h(t) = e^{-t}(1+t)^{-1}$, and following the discussion presented above we find that

$$\lambda(t,s) = \frac{1}{1+s}\left(s - te^{t-s}\right).$$

Therefore, the iteration formula is given by

$$u_{n+1}(t) = u_n(t) + \int_0^t \left[\frac{s - te^{t-s}}{1+s}\left(u_n''(s) - \frac{2+s}{1+s}u_n'(s) + u_n^2(s) - s^2e^{2s}\right)\right] ds.$$

We start with initial approximation $u_0(t) = t$. This in turn gives the successive approximations

$$\begin{aligned}
u_1(t) &= t + t^2 + \frac{t^3}{2} + \frac{t^4}{6} + \frac{17t^5}{120} + \frac{13t^6}{120} + \frac{251t^7}{5040} + \frac{113t^8}{5040} + \ldots \\
u_2(t) &= t + t^2 + \frac{t^3}{2} + \frac{t^4}{6} + \frac{t^5}{24} + \frac{t^6}{120} + \frac{t^7}{720} - \frac{17t^8}{5040} + \ldots \\
u_3(t) &= t + t^2 + \frac{t^3}{2} + \frac{t^4}{6} + \frac{t^5}{24} + \frac{t^6}{120} + \frac{t^7}{720} + \frac{t^8}{5040} + \ldots \\
&\vdots
\end{aligned}$$

When the iteration step n tends to infinity, this will yield the exact solution $u(t) = te^t$.

Example 4.2 *Consider the nonlinear Lane-Emden type equation*

$$u''(t) + \frac{2}{t}u'(t) + 8e^{u(t)} + 4e^{\frac{u(t)}{2}} = 0$$

with initial conditions $u(0) = 0, u'(0) = 0$. The exact solution of this problem is $u(t) = -2\ln(1+t^2)$. Since $h(t) = t^2$, and following the discussion presented above we find that

$$\lambda(t,s) = \frac{s^2}{t} - s.$$

Therefore, the iteration formula is given by

$$u_{n+1}(t) = u_n(t) + \int_0^t \left[\left(\frac{s^2}{t} - s\right)\left(u_n''(s) + \frac{2}{s}u_n'(s) + 8e^{u_n(s)} + 4e^{\frac{u_n(s)}{2}}\right)\right] ds.$$

We start with initial approximation $u_0(t) = 0$. *This in turn gives the successive approximations*

$$\begin{aligned} u_1(t) &= -2t^2 + t^4 - \frac{3t^6}{7} + \frac{17t^8}{108} + ... \\ u_2(t) &= -2t^2 + t^4 - \frac{2t^6}{3} + \frac{353t^8}{756} + ... \\ u_3(t) &= -2t^2 + t^4 - \frac{2t^6}{3} + \frac{t^8}{2} + ... \\ &\vdots \\ u_n(t) &= -2\left(t^2 - \frac{t^4}{2} + \frac{t^6}{3} - \frac{t^8}{4} + ...\right) \end{aligned}$$

Recall that the exact solution is given by $u(t) = \lim_{n\to\infty} u_n(t)$. *This is in turn gives the exact solution* $u(t) = -2\ln(1+t^2)$. *The reader can compare the above result with [12].*

Example 4.3 *Consider the nonlinear Bratu type equation*

$$u''(t) - 2e^{u(t)} = 0$$

with initial conditions $u(0) = 0, u'(0) = 0$. *The exact solution of this problem is* $u(t) = -2\ln\cos t$. *Since* $h(t) = c$, *and following the discussion presented above we find that*

$$\lambda(t, s) = s - t.$$

Therefore, the iteration formula is given by

$$u_{n+1}(t) = u_n(t) + \int_0^t \left[(s-t)\left(u_n''(s) - 2e^{u_n(s)}\right)\right] ds.$$

We start with initial approximation $u_0(t) = 0$. *This in turn gives the successive approximations*

$$\begin{aligned} u_1(t) &= t^2 \\ u_2(t) &= t^2 + \frac{t^4}{6} + \frac{t^6}{30} + \frac{t^8}{168} + ... \\ u_3(t) &= t^2 + \frac{t^4}{6} + \frac{2t^6}{45} + \frac{11t^8}{840} + ... \\ u_4(t) &= t^2 + \frac{t^4}{6} + \frac{2t^6}{45} + \frac{17t^8}{1260} + ... \\ &\vdots \end{aligned}$$

Consequently the exact solution is given by $u(t) = -2\ln\cos t$. *This is the same result as in [13].*

Example 4.4 *Consider the nonlinear Duffing type equation*

$$u''(t) + 3u(t) - 2u^3(t) = \cos t \sin 2t$$

with initial conditions $u(0) = 0, u'(0) = 1$. *The exact solution of this problem is* $u(t) = \sin t$. *Since* $h(t) = c$, *and following the discussion presented above we find that*

$$\lambda(t, s) = s - t.$$

Therefore, the iteration formula is given by

$$u_{n+1}(t) = u_n(t) + \int_0^t \left[(s-t)\left(u_n''(s) + 3u_n(s) - 2u_n^3(s) - \cos s \sin 2s\right)\right] ds.$$

We start with initial approximation $u_0(t) = t$. *This in turn gives the successive approximations*

$$\begin{aligned}
u_1(t) &= t - \frac{t^3}{6} - \frac{t^5}{60} + \frac{61t^7}{2520} + ... \\
u_2(t) &= t - \frac{t^3}{6} + \frac{t^5}{120} + \frac{t^7}{630} + ... \\
u_3(t) &= t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + ... \\
&\vdots \\
u_n(t) &= t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + ...
\end{aligned}$$

Since $u(t) = \lim_{n\to\infty} u_n(t)$, *we get* $u(t) = \sin t$ *which is the same solution as obtained in [14].*

5 Conclusion

In this study, the applicability of VIM for obtaining solutions of a class of IVPs is demostrated with the most common nonlinear problems in mathematical physics. The method yields solutions in the forms of convergent series with easily calculable terms. Numerical examples show that the use of the VIM may result in exact solutions by a few iterations. It can be concluded that the VIM is a very powerful and easy tool for solving nonlinear IVPs.

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