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On Subclasses of Uniformly Convex Spirallike Functions and Corresponding Class of Spirallike Functions

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Abstract

We determine a sufficient condition for a function $f(z)$ to be uniformly convex spirallike of order α that is also necessary when $f(z)$ has negative coefficients. Similar results are also obtained for corresponding classes of spirallike functions.

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1 Introduction

Let $\mathcal{A}$ denote the class of all functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ defined on the unit disk $E = \{z : |z| < 1\}$ normalized by $f(0) = 0$, $f'(0) = 1$. Further, by $S_p(\alpha)$ we shall denote the class of spirallike function $f(z)$ in C and such that

$$Re\left(e^{-i\alpha}\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in E$$

and for some α with $|\alpha| < \pi/2$.

The function $f(z)$ is convex spirallike if $zf'(z)$ is spirallike. The function $f(z)$ is uniformly α-spirallike if the image of every circular arc Γz with centre at ζ lying in E is α-spirallike with respect to $f(\zeta)$. The class of all uniformly α-spirallike functions is denoted by $USP(\alpha)$. The function $f(z)$ is uniformly convex α-spiral if the image of every circular arc Γz with centre at ζ lying in E is convex α-spirallike. The class of all uniformly convex α-spiral functions is denoted by $UCSP(\alpha)$ [3]. In [3] the author obtained the analytic characterization for functions f in $USP(\alpha)$ and $UCSP(\alpha)$ respectively as follows:

$$f \in USP(\alpha) \Leftrightarrow Re\left\{\frac{e^{-i\alpha}(z-\zeta)f'(z)}{f(z)-f(\zeta)}\right\} \geq 0, z \neq \zeta, z, \zeta \in E \tag{1}$$

$$f \in UCSP(\alpha) \Leftrightarrow Re\left\{e^{-i\alpha}\left(1+\frac{(z-\zeta)f''(z)}{f'(z)}\right)\right\} \geq 0, z \neq \zeta, z, \zeta \in E, |\alpha| < \pi/2 \tag{2}$$

The one variable characterization for these classes is proved in [3].

Theorem 1.1 *Let $f \in \mathcal{A}$. Then $f \in UCSP(\alpha)$ if and only if*

$$Re\left\{e^{-i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|, z \in E$$

The class of functions $F(z) = zf'(z)$, $f(z) \in UCSP(\alpha)$ is a subclass of the spirallike functions and we denote it by $SP_p(\alpha)$. In fact, the function $f(z) \in \mathcal{A}$ is in $SP_p(\alpha)$ if and only if

$$Re\left\{e^{-i\alpha}\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{zf'(z)}{f(z)}-1\right|, z \in E$$

This condition is equivalent to

$$Re\left\{\frac{e^{-i\alpha}\frac{zf'(z)}{f(z)}+i\sin\alpha}{\cos\alpha}\right\} \geq \left|\frac{e^{i\alpha}\frac{zf'(z)}{f(z)}+i\sin\alpha}{\cos\alpha}-1\right|, |\alpha| < \pi/2.$$

For $\alpha = 0$ the classes $UCSP(\alpha)$ and $SP_p(\alpha)$ respectively reduces to the classes UCV and S_P introduced and studied by Ronning [5].

2 Main Results

Definition 2.1 *Let $UCSP(\alpha, \beta)$ be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, which satisfy the condition*

$$Re\left\{e^{-i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right| + \beta, \quad 0 \leq \beta < 1.$$

In what follows we give a sufficient condition for a function f to be in $UCSP(\alpha,\beta)$.

Theorem 2.1 *If* $\sum_{n=2}^{\infty}(2n-\cos\alpha-\beta)n|a_n|\leq\cos\alpha-\beta$ *then* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, $z\in E$ *is in* $UCSP(\alpha,\beta)$.

Proof By the Definition of $UCSP(\alpha,\beta)$ it is sufficient if we verify the condition

$$\left|\frac{zf''(z)}{f'(z)}\right|\leq Re\ e^{-i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)-\beta.$$

That is, if $\left|\frac{zf''(z)}{f'(z)}\right|-Re\ e^{-i\alpha}\frac{zf''(z)}{f'(z)}\leq cos\alpha-\beta.$
We have

$$\begin{aligned}\left|\frac{zf''(z)}{f'(z)}\right|-Re\ e^{-i\alpha}\frac{zf''(z)}{f'(z)} &\leq 2\left|\frac{zf''(z)}{f'(z)}\right|\\ &\leq \frac{2\sum_{n=2}^{\infty}n(n-1)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}n|a_n||z|^{n-1}}\\ &\leq \frac{2\sum_{n=2}^{\infty}n(n-1)|a_n|}{1-\sum_{n=2}^{\infty}n|a_n|}\end{aligned}$$

$$\leq\frac{\sum_{n=2}^{\infty}(2n-\cos\alpha-\beta)n|a_n|}{1-\sum_{n=2}^{\infty}n|a_n|}$$

$$\therefore\quad \left|\frac{zf''(z)}{f'(z)}\right|-Re\ e^{-i\alpha}\left(\frac{zf''(z)}{f'(z)}\right)\leq\cos\alpha-\beta$$

only if $\sum_{n=2}^{\infty}(2n-\cos\alpha-\beta)n|a_n|\leq\cos\alpha-\beta.$ □

Definition 2.2 *Let $UCSPT(\alpha,\beta)$ be the class of functions $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ which satisfy the condition*

$$Re\ e^{-i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \left|\frac{zf''(z)}{f'(z)}\right| + \beta.$$

Theorem 2.2 *Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$. Then $\sum_{n=2}^{\infty}(2n - \cos\alpha - \beta)na_n \leq \cos\alpha - \beta$ if and only if $f(z)$ is in $UCSPT(\alpha,\beta)$.*

Proof In view of Theorem 2.1 we need only show that $f(z)$ is in $UCSPT(\alpha,\beta)$ satisfies the coefficient inequality.

If $f(z) \in UCSPT(\alpha,\beta)$ and z is real then the Definition of $UCSPT(\alpha,\beta)$ gives,

$$\cos\alpha - \frac{\cos\alpha \sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} na_n z^{n-1}} - \beta \geq \frac{\sum_{n=2}^{\infty} n(n-1)a_n z^n}{1 - \sum_{n=2}^{\infty} na_n z^{n-1}}$$

Let $z \to 1$ along the real axis, then we get

$$\cos\alpha - \beta \geq (1 + \cos\alpha)\frac{\sum_{n=2}^{\infty} n(n-1)a_n}{1 - \sum_{n=2}^{\infty} na_n}$$

$$(\cos\alpha - \beta)\left(1 - \sum_{n=2}^{\infty} na_n\right) \geq (1 + \cos\alpha)\sum_{n=2}^{\infty} n(n-1)a_n$$

$$\Rightarrow \cos\alpha - \beta \geq \sum_{n=2}^{\infty}(1 + \cos\alpha)(n-1)na_n$$

$$\Rightarrow \cos\alpha - \beta \geq \sum_{n=2}^{\infty}(2n - \cos\alpha - \beta)na_n$$

which gives the required result. □

Definition 2.3 *A function $f(z)$ is in $SP_p(\alpha,\beta)$ if $f(z)$ satisfies the analytic characterization*

$$Re\ e^{-i\alpha}\frac{zf'(z)}{f(z)} \geq \left|\frac{zf'(z)}{f(z)} - 1\right| + \beta$$

$\alpha \leq 1$, $\beta \geq 0$, *when* $\beta = 0$, $SP_p(\alpha,\beta)$ *becomes* $SP_p(\alpha)$.

Remark 2.1 $f(z) \in UCSP(\alpha)$ *if and only if* $zf' \in SP_p(\alpha)$.

Theorem 2.3 *If* $\sum_{n=2}^{\infty}(2n-\cos\alpha)n|a_n| \leq \cos\alpha$ *then* $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$, $z \in E$ *is in* $UCSP(\alpha)$.

Proof When $\beta = 0$ in Theorem 2.1 we get Theorem 2.3. □

Theorem 2.4 *If* $\sum_{n=2}^{\infty}(2n - \cos\alpha)|a_n| \leq \cos\alpha$ *then* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *is in* $SP_p(\alpha)$.

Proof By Alexander type Theorem we get that $f(z) \in UCSP(\alpha)$ if and only if $zf'(z) \in SP_p(\alpha)$.

Replacing the coefficient $|a_n|$ in Theorem 2.3 by $\frac{|a_n|}{n}$ we get the required result. □

Remark 2.2 $f(z) \in UCSP(\alpha,\beta)$ *if and only if* $zf' \in SP_p(\alpha,\beta)$.

Theorem 2.5 *If* $\sum_{n=2}^{\infty}(2n-\cos\alpha-\beta)|a_n| \leq \cos\alpha-\beta$ *then* $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$, $z \in E$ *is in* $SP_p(\alpha,\beta)$.

Proof By Alexander type Theorem $f(z) \in UCSP(\alpha,\beta)$ if and only if $zf'(z) \in SP_p(\alpha,\beta)$.

Hence replacing $|a_n|$ in Theorem 2.1 by $\frac{|a_n|}{n}$ we get the result.

Since $f(z) \in UCSPT(\alpha,\beta)$ if and only if $zf'(z) \in SP_pT(\alpha,\beta)$ the coefficient a_n in Theorem 2.2 can be replaced by $\frac{a_n}{n}$ to get the result for $SP_pT(\alpha,\beta)$. □

Theorem 2.6 $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$ *is in* $SP_pT(\alpha,\beta)$ *if and only if* $\sum_{n=2}^{\infty}(2n - \cos\alpha - \beta)a_n \leq \cos\alpha - \beta$.

3 Convolution Theorems

Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$ and $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$, $b_n \geq 0$. We investigate the nature of quasi-convolution $h(z) = f(z) * g(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$, given that $f(z)$ and $g(z)$ are members of subclasses of $UCSP(\alpha, \beta)$ and $SP_p(\alpha, \beta)$.

Theorem 3.1 *If $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$ and $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$, $b_n \geq 0$ are elements of $SP_pT(\alpha, \beta)$ then $(f * g)(z) = h(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$ in $SP_pT(\alpha, r)$ where*

$$r = \frac{\cos\alpha[8 + \cos^2\alpha + \beta^2 - 6\cos\alpha + \beta\cos\alpha - \beta] - 2\beta^2}{2(4 - 2\cos\alpha - 2\beta + \beta\cos\alpha)}$$

$0 \leq \alpha < 1$, $\beta \geq 0$. *The result is best possible.*

Proof Since $f(z)$ and $g(z)$ are in $SP_pT(\alpha, \beta)$ we have $\sum_{n=2}^{\infty}(2n - \cos\alpha - \beta)a_n \leq \cos\alpha - \beta$ and $\sum_{n=2}^{\infty}(2n - \cos\alpha - \beta)b_n \leq \cos\alpha - \beta$. We wish to find the larget r such that $\sum_{n=2}^{\infty}(2n - \cos\alpha - r)a_n b_n \leq \cos\alpha - r$. Equivalently we want to show that the conditions

$$\sum_{n=2}^{\infty} \frac{2n - \cos\alpha - \beta}{\cos\alpha - \beta} a_n \leq 1 \tag{3}$$

and

$$\sum_{n=2}^{\infty} \frac{2n - \cos\alpha - \beta}{\cos\alpha - \beta} b_n \leq 1 \tag{4}$$

imply that

$$\sum_{n=2}^{\infty} \frac{2n - \cos\alpha - r}{\cos\alpha - r} a_n b_n \leq 1 \tag{5}$$

for all

$$r \leq \frac{\cos\alpha[8 + \cos^2\alpha + \beta^2 - 6\cos\alpha + \beta\cos\alpha - \beta] - 2\beta^2}{2(4 - 2\cos\alpha - 2\beta + \beta\cos\alpha)}$$

From (3) and (4) and by means of Cauchy Schwarz inequality, we get that

$$\sum_{n=2}^{\infty} \frac{2n - \cos\alpha - \beta}{\cos\alpha - \beta} \sqrt{a_n b_n} \leq 1 \tag{6}$$

Hence it is enough if we prove

$$\frac{2n - \cos\alpha - r}{\cos\alpha - r} a_n b_n \leq \frac{2n - \cos\alpha - \beta}{\cos\alpha - \beta} \sqrt{a_n b_n}$$
$$r \leq r(\alpha, \beta), \quad n = 2, 3, \ldots$$

(or)

$$\sqrt{a_n b_n} \leq \left(\frac{2n - \cos\alpha - \beta}{\cos\alpha - \beta}\right) \left(\frac{\cos\alpha - r}{2n - \cos\alpha - r}\right)$$

From (6) it follows that

$$\sqrt{a_n b_n} \leq \frac{\cos\alpha - \beta}{2n - \cos\alpha - \beta} \quad \text{for all } n \tag{7}$$

The above inequality is equivalent to

$$\frac{r + \cos\alpha}{2} \leq \frac{\cos\alpha - n\left[\frac{\cos\alpha - \beta}{2n - \cos\alpha - \beta}\right]^2}{1 - \left[\frac{\cos\alpha - \beta}{2n - \cos\alpha - \beta}\right]^2} \tag{8}$$

The right hand side of (8) is an increasing function of n, $(n = 2, 3, \ldots)$. By taking $n = 2$ in (8) we get

$$r \leq \frac{\cos\alpha[8 + \cos^2\alpha + \beta^2 - 6\cos\alpha + \beta\cos\alpha - \beta] - 2\beta^2}{2(4 - 2\cos\alpha - 2\beta + \beta\cos\alpha)}$$

The result is sharp with equality where

$$f(z) = g(z) = z - \frac{\cos\alpha - \beta}{4 - \cos\alpha - \beta}\ z^2$$

□

Corollary 3.1 *For $f(z)$ and $g(z)$ as in Theorem 3.1 we have*

$$h(z) = z - \sum_{n=2}^{\infty} \sqrt{a_n b_n} z^n \in SP_pT(\alpha, \beta)$$

Proof The result follows from Cauchy-Schwarz inequality and (6). The result is sharp for the same function in Theorem 3.1. □

Theorem 3.2 *For $f(z) \in SP_pT(\alpha, \beta_1)$ and $g(z) \in SP_pT(\alpha, \beta_2)$ we have $f(z) * g(z) \in SP_pT(\alpha, r)$ where*

$$r \leq \frac{\cos\alpha(8 + \cos^2\alpha + \beta_1\beta_2 - 6\cos\alpha) - 2\beta_1\beta_2}{8 - 4\cos\alpha - 2(\beta_1 + \beta_2) + (\beta_1 + \beta_2)\cos\alpha}$$

Proof Proceeding as in the proof of Theorem 3.1 we get

$$\frac{\cos\alpha + r}{2} \leq \frac{\cos\alpha - n\left[\frac{\cos\alpha - \beta_1}{2n - \cos\alpha - \beta_1}\right]\left[\frac{\cos\alpha - \beta_2}{2n - \cos\alpha - \beta_2}\right]}{1 - \left(\frac{\cos\alpha - \beta_1}{2n - \cos\alpha - \beta_1}\right)\left(\frac{\cos\alpha - \beta_2}{2n - \cos\alpha - \beta_2}\right)} \tag{9}$$

The right hand side of (9) is an increasing function of $n = 2, 3, \ldots$. Setting $n = 2$ we get

$$r \leq \frac{\cos\alpha(8 + \cos^2\alpha + \beta_1\beta_2 - 6\cos\alpha) - 2\beta_1\beta_2}{8 - 4\cos\alpha - 2(\beta_1 + \beta_2) + (\beta_1 + \beta_2)\cos\alpha}$$

□

Corollary 3.2 *Let $f(z), g(z), h(z) \in SP_pT(\alpha, \beta)$. Then $f(z) * g(z) * h(z) \in SP_pT(\alpha, r_1)$ where*

$$r_1 \leq \frac{\begin{array}{c}4\beta^3 + \cos^4\alpha(3\beta - 4) + \cos^3\alpha(32 + \beta^2)\\ + \cos^2\alpha(\beta^3 - 3\beta^2 + 60\beta - 80) + \cos\alpha(-4\beta^3 + 2\beta^2 - 48\beta + 64)\end{array}}{\begin{array}{c}64 + 8\beta^2 - 48\beta + \cos^4\alpha + \cos^3\alpha(\beta - 8)\\ + \cos^2\alpha(36 - 15\beta + 3\beta^2) + \cos\alpha(80 + 50\beta - 12\beta^2)\end{array}}$$

Proof From Theorem 3.1 we get $f(z) * g(z) \in SP_pT(\alpha, r)$ where

$$r \leq \frac{\cos\alpha[8 + \cos^2\alpha + \beta^2 - 6\cos\alpha + \beta\cos\alpha - \beta] - 2\beta^2}{2(4 - 2\cos\alpha - 2\beta + \beta\cos\alpha)}$$

$f(z) * g(z) * h(z) \in SP_pT(\alpha, r_1)$ where

$$r_1 \leq \frac{\cos\alpha(8 + \cos^2\alpha + \beta r - 6\cos\alpha) - 2\beta r}{8 - 4\cos\alpha - 2\beta - 2r + (\beta + r)\cos\alpha}$$

substituting for r and simplifying we get the required result. □

Theorem 3.3 *Let* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$ *and* $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$, $b_n \geq 0$ *be elements of* $UCSPT(\alpha, \beta)$ *then*

$$f(z) * g(z) = h(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n \in UCSPT(\alpha, r)$$

where

$$r \leq \frac{3\cos^3\alpha + \cos^2\alpha(2\beta - 20) + \cos\alpha(32 + 3\beta^2 - 8\beta) - 4\beta^2}{32 + \beta^2 - 16\beta + \cos^2\alpha + \cos\alpha(6\beta - 16)}$$

Proof From Theorem 2.2 we have for $f(z) \in UCSPT(\alpha, \beta)$.

$$\sum_{n=2}^{\infty}(2n - \cos\alpha - \beta)na_n \leq \cos\alpha - \beta$$

$g(z) \in UCSPT(\alpha, \beta)$

$$\sum_{n=2}^{\infty}(2n - \cos\alpha - \beta)nb_n \leq \cos\alpha - \beta$$

Hence proceeding as in Theorem 3.1 we want to get a r which satisfies

$$\sum_{n=2}^{\infty} \frac{n(2n - \cos\alpha - r)a_n b_n}{\cos\alpha - r} \leq 1$$

Using the same method we get

$$\frac{r + \cos\alpha}{2} \leq \frac{\cos\alpha - \left[\frac{\cos\alpha - \beta}{2n - \cos\alpha - \beta}\right]^2}{1 - \frac{1}{n}\left[\frac{\cos\alpha - \beta}{2n - \cos\alpha - \beta}\right]^2}$$

This is an increasing function of n $(n = 2, 3, \ldots)$.

By setting $n = 2$ we get

$$\frac{r + \cos\alpha}{2} \leq \frac{\cos\alpha - 2\left[\frac{\cos\alpha - \beta}{4 - \cos\alpha - \beta}\right]^2}{2 - \left[\frac{\cos\alpha - \beta}{4 - \cos\alpha - \beta}\right]^2}$$

or

$$r \leq \frac{3\cos^3\alpha + \cos^2\alpha(2\beta - 20) + \cos\alpha(32 + 3\beta^2 - 8\beta) - 4\beta^2}{32 + \beta^2 - 16\beta + \cos^2\alpha + \cos\alpha(6\beta - 16)}$$

□

Theorem 3.4 *Let $f(z) \in UCSPT(\alpha,\beta_1)$ and $g(z) \in UCSPT(\alpha,\beta_2)$ then $f(z) * g(z) \in UCSPT(\alpha, r)$ where*

$$r \leq \frac{\cos\alpha[32-4(\beta_1+\beta_2)+3\beta_1\beta_2]+\cos^2\alpha(\beta_1+\beta_2-20)+3\cos^3\alpha-4\beta_1\beta_2}{32-\cos\alpha(16-3(\beta_1+\beta_2))+\cos^2\alpha-8(\beta_1+\beta_2)+\beta_1\beta_2}$$

Proceeding as in Theorem 3.3 we get

$$\frac{r+\cos\alpha}{2} \leq \frac{\cos\alpha-\left[\frac{\cos\alpha-\beta_1}{2n-\cos\alpha-\beta_1}\right]\left[\frac{\cos\alpha-\beta_2}{2n-\cos\alpha-\beta_2}\right]}{1-\frac{1}{n}\left(\frac{\cos\alpha-\beta_1}{2n-\cos\alpha-\beta_1}\right)\left(\frac{\cos\alpha-\beta_2}{2n-\cos\alpha-\beta_2}\right)}$$

Right hand side is an increasing function for $n = 2, 3, \ldots$. Taking $n = 2$ we get the required result.

$$r \leq \frac{\cos\alpha[32-4(\beta_1+\beta_2)+3\beta_1\beta_2]+\cos^2\alpha(\beta_1+\beta_2-20)+3\cos^3\alpha-4\beta_1\beta_2}{32-\cos\alpha(16-3(\beta_1+\beta_2))+\cos^2\alpha-8(\beta_1+\beta_2)+\beta_1\beta_2}$$

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