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Compact Weighted Composition Operators on a Space of Continuous Functions

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Abstract

Let $C_{co}(X,E)$ be the locally convex space of all continuous functions from a completely regular Hausdorff space X into a Banach space E,endowed with its compact-open topology.In this paper ,we characterize those operator-valued functions π on X and selfmaps T on X which induce compact weighted composition operators $W_{\pi,T}$ on $C_{co}(X,E)$.

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1 Introduction and Preliminaries

The content of this paper is in relation with the theory of weighted composition operators which are studied in [1,2,3,8,9,10,11,13]. For more details, we refer to [7] as an excellent recent source on the theory of composition operators.

Throughout this paper, E stands for a nontrivial (real or complex) Banach space and V is a system of weights on a completely regular Hausdorff space X, where by a weight on X we mean a nonnegative upper semi-continuous function on X and a family V of weights is said to be a system of weights on X if it is directed upward and also $V > 0$ [4]. By $B_S(E)$(respectively, $B_u(E)$) we

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denote the space $B(E)$ of all operators (i.e. continuous linear transformations) on E equipped with the strong (respectively, uniform) operator topology. Let $C(X, E)$ be the vector space of all continuous functions from X into E, and define the weighted spaces $CV_i(X, E)$ of E-valued continuous functions on X with respect to a given system V of weights as follows:
$CV_o(X, E) = \{f \in C(X, E) : vf \ \ vanishes \ \ at \ \ infinity \ \ on \ \ X \ \ for \ \ all \ \ v \in V\}$, and
$CV_b(X, E) = \{f \in C(X, E) : vf(X) \ \ is \ \ bounded \ \ in \ \ E \ \ for \ \ all \ \ v \ \ \in V\}$.
Then it is clear that $CV_i(X, E)$,where $i \in \{o, b\}$, is a vector space with the pointwise linear operations, while the upper semi-continuity of weights implies that $CV_o(X, E) \subset CV_b(X, E)$. For $v \in V$ and $f \in C(X, E)$, if we put $P_v(f) = \sup\{v(X)||f(X)|| : x \in X\}$, then P_v is a seminorm on $CV_b(X, E)$ (and hence on $CV_0(X, E)$), and the family $\{P_v : v \in V\}$ of seminorms generates a Hausdorff locally convex topology on each of these spaces. This topology is called the weighted topology, and $CV_i(X, E)$ endowed with this topology is called the weighted space of vector-valued continuous functions.

It is known that when V is the system of all positive constant functions on X , $CV_i(X, E) = (C_i(X, E), u)$,where $i \in \{0, b\}$ and u denotes the topology of uniform convergence on X, where as for the system V of all weights on X which vanish at infinity,$CV_i(X, E) = (C_b(X, E), \beta_0)$ for $i \in \{o, b\}$, where β_0 denotes the substrict topology. Further, in case V is the system of all weights on X generated by characteristic functions of all compact subsets of X, it is known that $CV_i(X, E) = C_{co}(X, E)$ for $i \in \{o, b\}$, where $C_{co}(X, E)$ is the space $C(X, E)$ endowed with the compact-open topology. For details about these weighted spaces of continuous functions and for the notations not explained here, please see [4] or [7].

Suppose that $i \in \{o, b\}$. Let π be a $B(E)$- valued function on X and T be a self map on X such that $\pi \cdot f \circ T$ belongs to $CV_i(X, E)$ whenever $f \in CV_i(X, E)$, where the product of π with the composite function $f \circ T$ is defined pointwise on X. Then the map $f \to \pi \cdot f \circ T$ is a linear transformation on $CV_i(X, E)$. In case it is also continuous, we call it the weighted composition operator on $CV_i(X, E)$ induced by the pair (π, T), and designate it by the symbol $W_{\pi,T}$. Such operators on weighted spaces of continuous functions have been studied in [5,9,10,12].

Recall that a topological space Y is said to be a K_R- space if every real function on Y which is continuous on each compact subset of Y is continuous, and a linear transformation A on a topological vector space F is compact if the image of every bounded subset of F under A is relatively compact in F. A subset H of $C(X, E)$ is equicontinuous on X if and only if for every x in X and for every net $x_\alpha \to x$ in X implies that

$$\sup\{||h(x_{(\alpha)} - h(X)|| : h \in H\} \to 0.$$

The following compactness criteria of Arzela-Ascoli type from [4] will be needed in the characterization of compact weighted composition operators on $C_{co}(X, E)$:
Proposition A. Assume that X is also a K_R- space. Then a subset H of $C_{co}(X, E)$ is relatively compact if and only if
(i) H is equicontinuous on X, and
(ii) $H(X) = h(X) : h \in H$ is relatively compact in E for all x in X.

WEIGHTED COMPOSITION OPERATORS

In this section, we shall characterize those functions $\pi : X \to B(E)$ and $T : X \to X$ which induce weighted composition operators on $C_{co}(X, E)$. In [10,Theorem 3.2], a characterization of weighted composition operator $W_{\pi,T}$ on $CV_b(X, E)$ has been obtained by Singh and one of the authors. This theorem in our setting of $C_{co}(X, E)$ yields the following :
Proposition 1. Let T be a continuous self map on X and let $\pi \in C(X, B_s(E))$ such that $\pi(X)$ is an equicontinuous subset of B(E). Then $W_{\pi,T}$ is a weighted composition operator on $C_{co}(X, E)$.
In the next theorem, we will attempt to improve Proposition 1. First, let us note the following facts about a weighted composition operator $W_{\pi,T}$ on $C_{co}(X, E)$:
For each $e \in E$ let 1_e be the constant e-function on X, that is, $l_e(x) = e$ for all $x \in X$. Then it is clear that $l_e \in C_{co}(X, E)$, and $W_{\pi,T^1e}(x) = \pi(x)e$ for $x \in X$. Moreover, if $\{x_\alpha\}$ is a net in X with $x_\alpha \to x_0$ in X, then we have

$$\|\pi(x_\alpha)e - \pi(x_0)e\| = \|W_{\pi,T^1e}(x_\alpha) - W_{\pi,T^1e}(x_0)\| \to 0,$$

since $W_{\pi,T^1e} \in C(X, E) and x - \alpha \to x_0$. This implies that the map $\pi : X \to B_s(E)$ is continuous in the strong operator topology, that is, $\pi \in C(X, B_s(E))$. (Note that π is not necessarily continuous in the uniform operator topology. See [2]for example). This continuity of π shows that $N(\pi) = \{x \in X : \pi(x) \neq 0\}$ is open in X. Also, we shall see that T is continuous on $N(\pi)$. But T is not necessarily continuous on $X - N(\pi)$, because, for every

$f \in C_{co}(X, E), W_{\pi,T^f}$ is zero on $X - N(\pi)$ even if T is anyhow defined.

Theorem 2. Let π be a B(E)- valued function on X such that $\pi(X)$ is bounded and let T be a selfmap on x. Then $W_{\pi,T}$ is a weighted composition operator on $C_{co}(X, E)$ if and only if (2.1) $\pi \in C(X, B_s(E))$ and (2.2) T is continuous on $N(\pi)$.

Proof. Assume that (2.1) and (2.2) holds. Then it follows from Proposition l that $W_{\pi,T}$ is a weighted composition operator on $C_{co}(X, E)$. Conversely if $W_{\pi,T}$ is a weighted composition operator on $C_{co}(X, E)$, then, as noted above, (2.1) holds.

To prove (2.2), we suppose on the contrary that there exists an x_0 in $N(\pi)$ at which T is not continuous. Then there is net $\{x_\alpha : \alpha \in A\}$ in N_π such that $x_\alpha \to x_0$ but $T(x_\alpha)$ does not converge to $T(x_0)$. This further implies that there is an open neighbourhood U of $T(x_0)$ such that,for each $\alpha_0 \in A, T(x_\alpha) \notin U$ for some $\alpha \geq \alpha_0$. Thus we can find a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ such that $T(x_\beta) \notin U$. The complete regularity of X gives a continuous function f on X such that $0 \leq f \leq 1, f(T(x_0)) = 1$ and $f(X - U) = \{0\}$. Choose $e \in E$ such that $\pi(x_0)e \neq 0$, and define the function f_e as $f_e(x) = f(x)e$ for all x in X. Then it is clear that $f_e \in C_{co}(X, E)$. But for each β, we have

$$\|W_{\pi,T}f_e(x_\beta) - W_{\pi,T}f_e(x_0)\| = \|\pi(x_\beta)f_e(T(x_\beta)) - \pi(x_0)f_e(T(x_o))\|$$

$$= \|\pi(x_0)e\| > 0.$$

This implies that $W_{\pi,T}f_e \notin C_{co}(X, E)$, which contradicts the fact that $W_{\pi,T}$ is an operator on $C_{co}(X, E)$. Therefore T must be continuous on $N(\pi)$.

Remark. The theorem above corrects [10, Proposition 4.3].

Now, let us remark about multiplication operators on $C_{co}(X, E)$. Let T be the identity map on X. Then the corresponding operator $W_{\pi,T}$ on $C_{co}(X, E)$ is denoted by M_π and is called the multiplication operator induced by π. In this setting, our theorem yields the following corollary, which is a corrected version (cf. [6]) of [5, Proposition 2.2].

Corollary 3. Let π be a B(E)- valued function on X such that $\pi(X)$ is bounded. Then M_π is a multiplication operator on $C_{co}(X, E)$ if and only if $\pi \in C(X, B_s(E))$.

COMPACT WEIGHTED COMPOSITION OPERATORS

The purpose of this section is to study compact weighted composition operators on $C_{co}(X, E)$. When X is a compact Hausdorff space, $C_{co}(X, E) =$

$(C(X,E),u)$. In this setting compact weighted composition operators have been characterized by Jamison and Rajagopalan in [2], and their result has been improved by Takagi while working on a more general space of E-valued functions on X in [13]. Chan [1] has studied compactness of these operators on $(C_i(X,E),u)$, while in a recent work Singh and Manhas [8] have studied them on $(C_b(X,E),\beta_o)$. Also, in [9] one of the authors with Singh and Manhas have studied compact composition operators on weighted spaces under the assumptions that X is connected and V consists of bounded weights (or weights which vanish at infinity) on X, where as Lindstrom and Liovana have studied compact and weakly compact homomorphisms of composition type on the locally convex algebras $C_{co}(X,C)$ in [3].

In the following theorem, we now present a characterization of compact weighted composition operators on $C_{co}(X,E)$. This result is an improved version (cf. [13]) of [2, Theorem 2].

Theorem 4. Assume that X is also a K_R- space and let π be a B(E)- valued function on X and T be a selfmap on X. Then $W_{\pi,T}$ is a compact weighted composition operator on $C_{co}(X,E)$ if and only if the following conditions hold:
(4.1) $\pi \in C(X,B_u(E))$;
(4.2) T is continuous on $N(\pi)$;
(4.3) $\pi(x)$ is a compact operator on E for all x in X; and
(4.4) T is locally constant on $N(\pi)$.

Proof. Assume first that $W_{\pi,T}$ is a compact weighted composition operator on $C_{co}(X,E)$. Then the condition (4.2) follows from the continuity of the operator $W_{\pi,T}$, where as the proof of (4.1) is similar to that of the same part of [2, Theorem2]. To Prove (4.3), let $\{e_n\}$ be an arbitrary sequence in $B_1=\{e\in E : \|e\|\le 1\}$, and consider the sequence $\{l_{e_n}\}$ of constant e_n-functions, where $l_{e_n}(x)=e_n$ for all x in X. Then $\{l_{e_n}\}$ is a bounded sequence in $C_{co}(X,E)$, and so the compactness of $W_{\pi,T}$ implies that $H=\{W_{\pi,T}l_{e_n}\}$ is relatively compact in $C_{co}(X,E)$. But, according to proposition A, this implies that $H(x)=\{\pi(x)e_n\}$ is relatively compact in E for every $x\in X$. Thus each $\pi(x)$ is a compact operator on E.

To show (4.4), assume that there exists an x_o in $N(\pi)$ such that T is not constant on any open neighbourhood U of x_o. Let $\mathcal{N}$ denote any fixed neighbourhood base at x_o and direct $\mathcal{N}$ with the following order relation:

$$U_1 \le U_2 \ \ if \ \ and \ \ only \ \ if \ \ U_1 \subset U_2.$$

With the above assumption, there exists a net $\{X_U\}$ in $N(\pi)$ such that $x_U \to x_O$ but $T(x_U) \ne T(x_O)$ for all U in $\mathcal{N}$. For each U in $\mathcal{N}$, the complete regularity of X gives a continuous function f_U on X such that $0 \leq f_U \leq 1, f_U(T(x)) = 1$ *and* $f_U(T(x_U)) = 0$. Choose $e \in E$ such that $\pi(x_o)e \neq 0$, and define the function f_{Ue} in $C_{co}(X, E)$ by setting $f_{Ue}(x) = f_U(x)e$ for all x in X. Let $B = \{f_{Ue} : U \in \mathcal{N}\}$. Then B is bounded in $C_{co}(X, E)$. But for each U in $\mathcal{N}$, we have

$$\|W_{\pi,T}f_{Ue}(x_u) - W_{\pi,T}f_{Ue}(x_o)\| = \|\pi(x_o)e\| > 0,$$

which contradicts the equicontinuity of $W_{\pi,T}(B)$. Thus we obtain the condition (4.4).
Conversly, assume that conditions(4.1)- (4.4)hold. Conditions (4.1) and (4.2) yield continuity of the linear transformation $W_{\pi,T}$. To prove compactness of $W_{\pi,T}$, let B be an arbitrary bounded subset of $C_{co}(X, E)$ and put $H = W_{\pi,T}(B)$. Let $x \in X$ be fixed. Since $\{f(T(x)) : f \in B\}$ is bounded in E and by condition (4.3) each $\pi(x)$ is compact on E, it follows that $\{\pi(x)f(T(x)) : f \in B\} = H(x)$ is relatively compact in E for all x in X.

We next see that H is equicontinuous on X. Let x_o be an arbitrary point in $N(\pi)$, and let $\{x_\alpha\}$ be a net in X such that $x_\alpha \to x_o$. This implies for each neighbourhood U of x_o in $N(\pi)$, there is an $\alpha_o \in A$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_o$. Since T is constant on $U, T(x_\alpha) = T(x_o)$ for $\alpha \geq \alpha_o$. But for each f in B, we have

$$\|W_{\pi,T}f(x_\alpha) - W_{\pi,T}f(x_o)\| = \|\pi(x_\alpha)f(T(x_\alpha)) - \pi(x_o)f(T(x_o))\|$$

$$\leq \|\pi(x_\alpha) - \pi(x_o)\| . \|f(T(x_o))\| ,$$

for each $\alpha \geq \alpha_o$. Since the set $\{f(T(x_o)) : f \in B\}$ is bounded in E, we obtain, in view of condition (4.1), that

$$sup\{\left\|W_{\pi,T}f(x_\alpha) - W_{\pi,T}f(x_{(o)}\right\| : f \in B\} \to 0$$

as $x_\alpha \to x_o$. Therefore, H is equicontinuous on $N(\pi)$. Similarly applying condition (4.1) again, it can be seen that H is also equicontinuous at other points of X. Now, according to Proposition A, it follows that $H = W_{\pi,T}(B)$ is relatively compact in $C_{co}(X, E)$. Thus $W_{\pi,T}$ is compact, and the proof of the theorem is completed.
Remark: The above theorem extends part(1) of Theorem 2.1 of [11].

Next, let us remark about composition operators on $C_{co}(X, E)$. Let I_E be the identity operator on E, and define $\pi(x) = I_E$ for all x in X. Then the corresponding operator $W_{\pi,T}$ on $C_{co}(X,E)$, denoted by C_T, is called the composition operator induced by T. In case E is a finite dimensional Banach space, I_E is compact and so the map π satisfies the condition (4.3) in the Theorem 4. Hence we obtain the following corollary:

Corollary 5. Assume that X is also a K_R- space, and let T be a continuous selfmap on X. Then the composition operator C_T on $C_{co}(X,E)$ is compact if and only if (5.1) E is finite dimensional, and (5.2) T is locally constant.

Further when $E=C$, $C_{co}(X,C)$, is the locally convex algebra of all complex valued continuous functions on X with its compact- open topology, and the condition (5.1) in the above corollary is satisfied. Consequently,we obtain an analogue for [3,Proposition 3].

Finally when E is an infinite dimensional Banach space, I_E is non-compact and the map $\pi(x)=I_E$ for all $x\ in\ X$does not satisfy the condition (4.3) in the Theorem 4. Therefore, we have

Corollary 6 If E is an infinite dimensional Banach space, then no composition operator on $C_{co}(X,E)$ is compact.

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